GRADIENT OF A SCALAR FIELD

We will denote by $f\,(M)$ a real function of a point M in an area A. If A is two dimensional, then

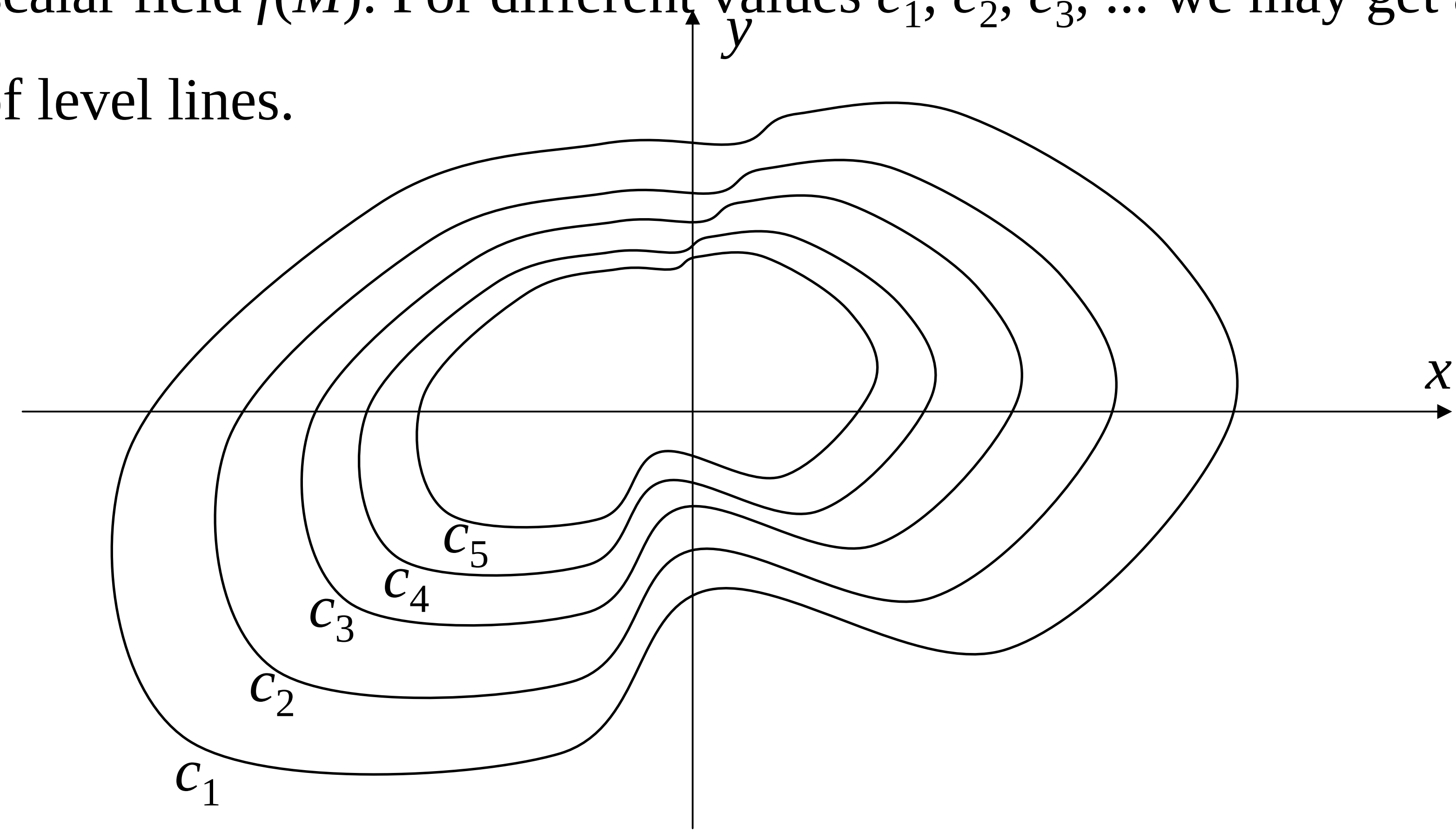
$$f(M) = f(x, y)$$

and, if A is a 3-D area, then

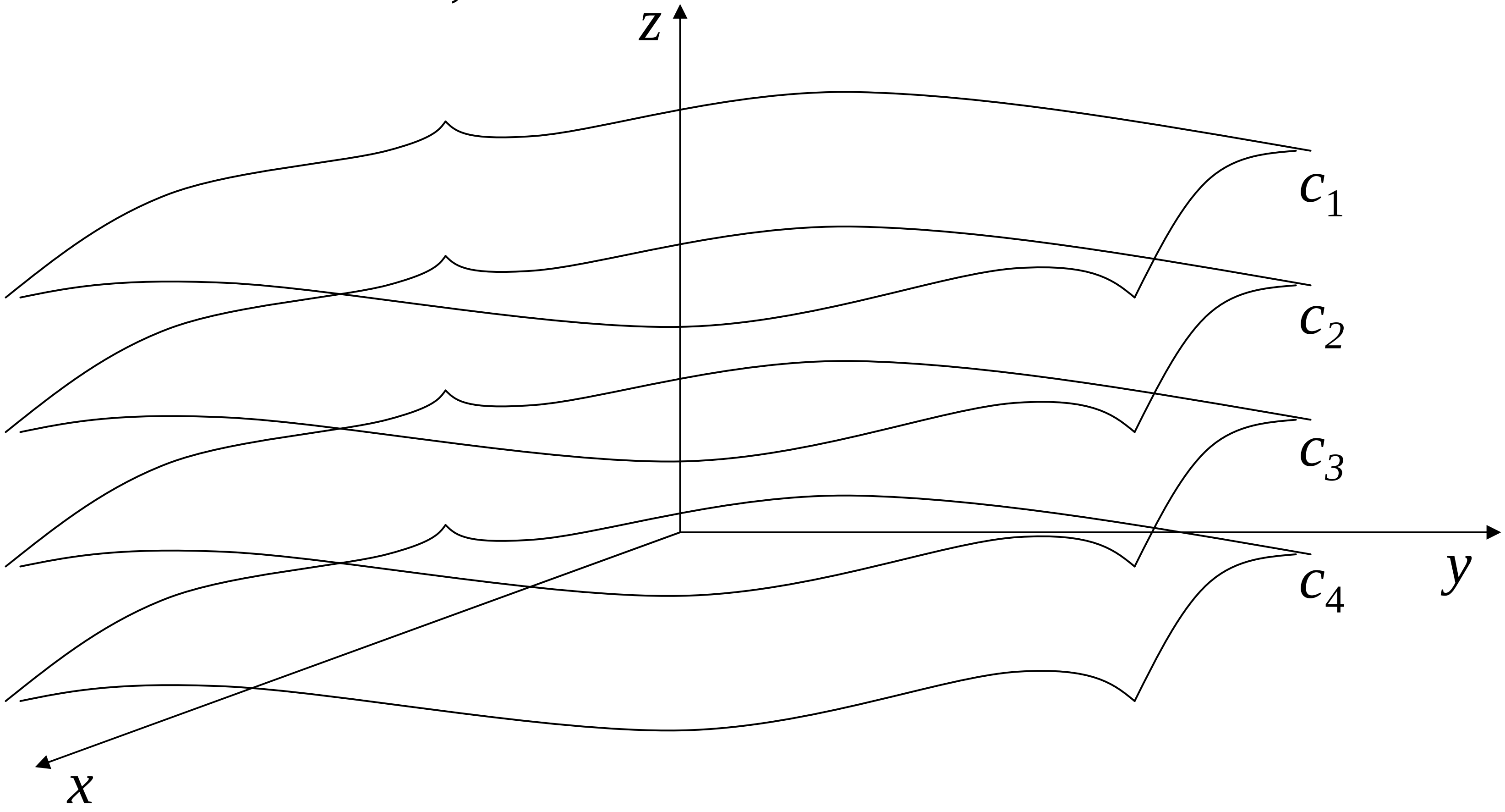
$$f(M) = f(x, y, z)$$

We will call $f\,(M)$ a ***scalar field*** defined in A.

Let $f(M) = f(x,y)$. Consider an equation $f(M) = c$. The curve defined by this equation is called a ***level line*** (contour line, height line) of the scalar field $f(M)$. For different values c_1, c_2, c_3, ... we may get a set of level lines.

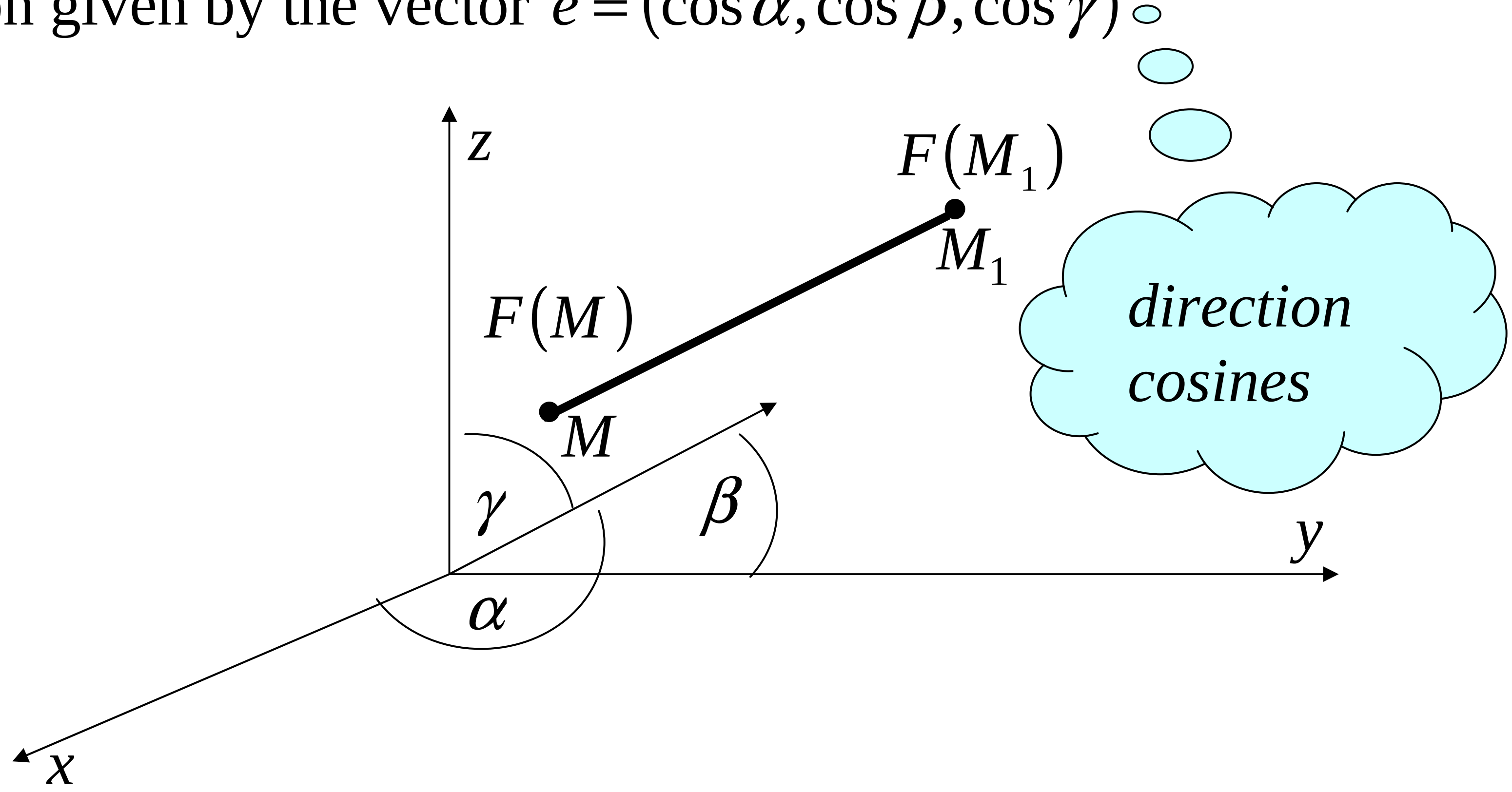


Similarly, if *A* is a 3-D area, the equation $f(M) = c$ defines a surface called a ***level surface*** (contour surface). Again, for different values of *c*, we will obtain a set of level surfaces.



Directional derivative

Let *F*(*M*) be a 3-D scalar field and let us construct a value that characterizes the rate of change of *F*(*M*) at a point *M* in a direction given by the vector $\vec{e} = (\cos\alpha, \cos\beta, \cos\gamma)$



$$\frac{\partial F}{\partial \vec{e}} = \lim_{M_1 \to M} \frac{F(M_1) - F(M)}{\overline{MM_1}}$$

Let $M = [x,y,z]$ and $M_1=[x+\Delta x,\ y+\Delta y,\ z+\Delta z\ ]$, then

$$F(M_1) - F(M) = F(x+\Delta x, y+\Delta y, z+\Delta z) - F(x,y,z) =$$

$$= dF(x,y,z) + \delta\rho = \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y + \frac{\partial F}{\partial z}\Delta z =$$

$$= \frac{\partial F}{\partial x}\rho\cos\alpha + \frac{\partial F}{\partial y}\rho\cos\beta + \frac{\partial F}{\partial z}\rho\cos\gamma + \delta\rho$$

where $\rho = \overline{MM_1}$ and $\delta \to 0$ as $\rho \to 0.$

This gives us

$$\frac{\partial F}{\partial \boldsymbol{e}} = \lim_{\rho \to 0} \left(\frac{\partial F}{\partial x} \cos\alpha + \frac{\partial F}{\partial y} \cos\beta + \frac{\partial F}{\partial z} \cos\gamma + \delta \right)$$

However, the first three terms of the limit do not depend on ρ and $\delta \to 0$ as $\rho \to 0$ so that

$$\frac{\partial F}{\partial \boldsymbol{e}} = \frac{\partial F}{\partial x} \cos\alpha + \frac{\partial F}{\partial y} \cos\beta + \frac{\partial F}{\partial z} \cos\gamma$$

Clearly, $\dfrac{\partial F}{\partial \vec{e}}$ assumes its greatest value for

$$\vec{e} = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k}$$

This vector is called the ***gradient*** of F denoted by

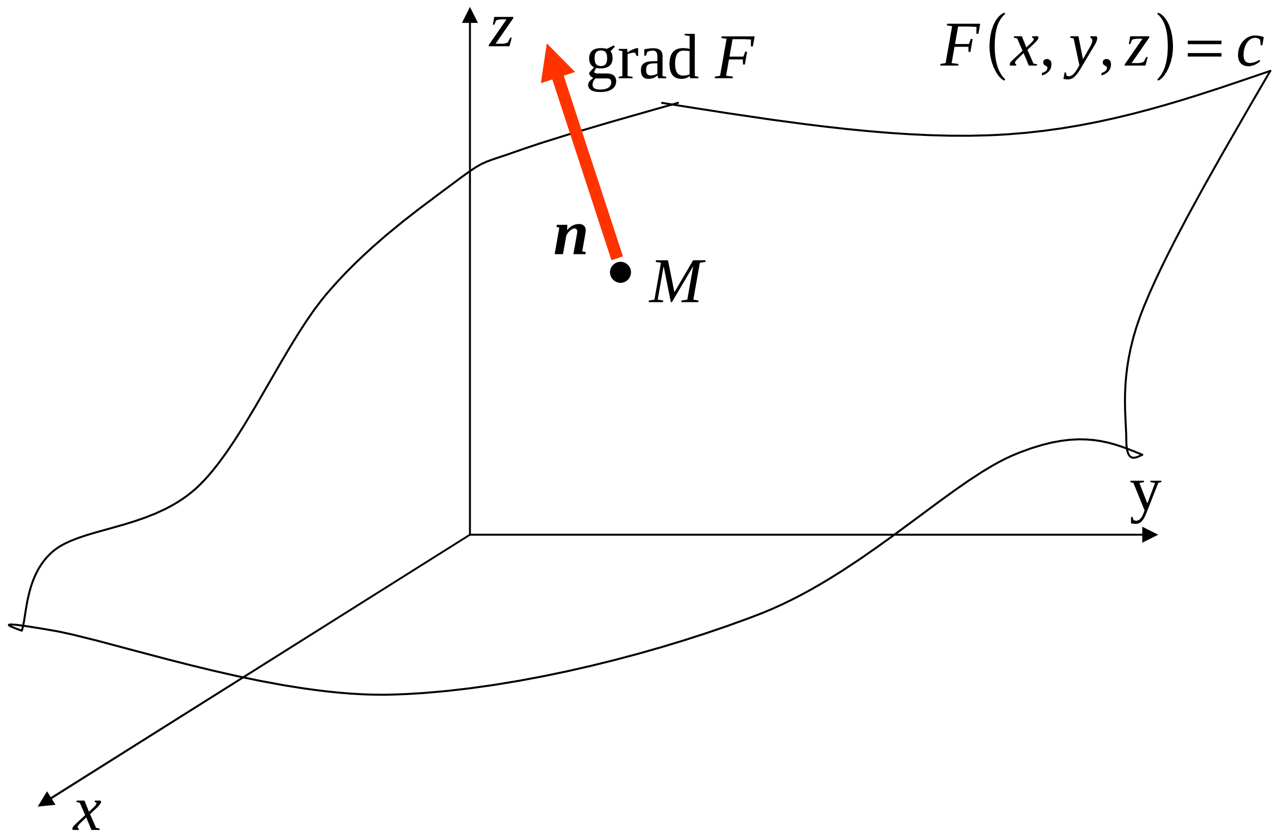
$$\operatorname{grad} F(x, y, z) = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k}$$

or, using the Hamiltonian operator ∇ (nabla):

$$\operatorname{grad} F = \nabla F$$

Thus grad F points in the direction of the steepest increase in F or the steepest slope of F. Geometrically, for a c, grad F at a point M is parallel the unit normal $\vec{n}$ at M of the level surface

$$F(x, y, z) = c$$

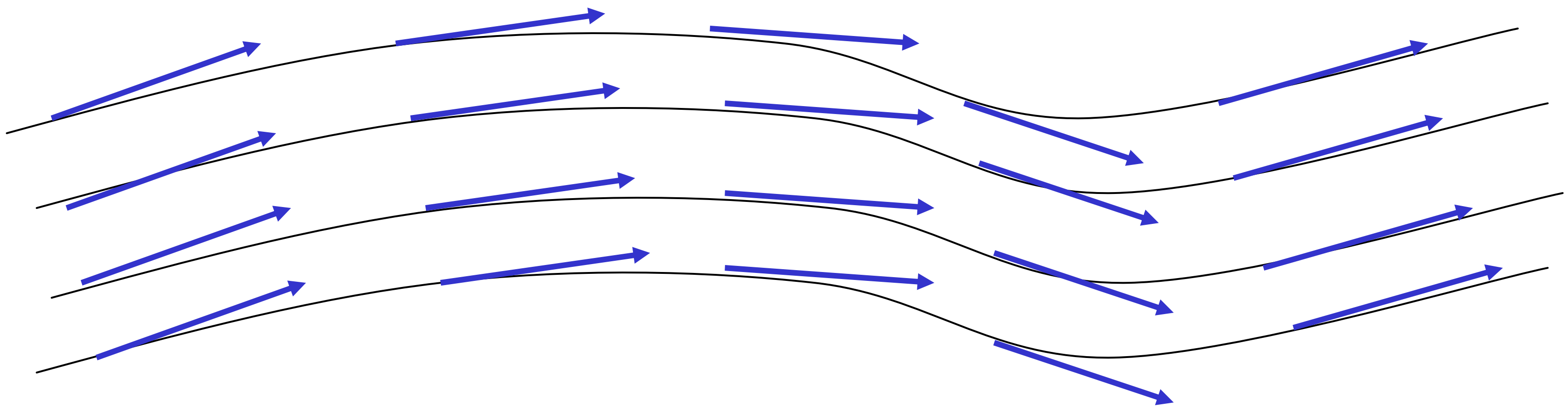


<u>Vector field, vector lines</u>

Let a vector field ***f*** (M) be given in a 3-D area **Ω,** that is, each $M \in \boldsymbol{\Omega}$ is assigned the vector

$$\vec{f}(M) = f_1(x, y, z)\vec{i} + f_2(x, y, z)\vec{j} + f_3(x, y, z)\vec{k}$$

A ***<u>vector line</u>*** *l* of the vector field $\vec{f}(M)$ is defined as a line with the property that the tangent vector to *l* at any point *L* of *l* is equal to $\vec{f}(L)$.



This means that if we denote by $d\vec{S} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$ the tangent vector of l, then, at each point M, the following equations hold $\dfrac{dx}{f_1(M)} = \dfrac{dy}{f_2(M)} = \dfrac{dz}{f_3(M)}$

This yields two differential equations defining the vector lines for $\vec{f}(M)$. This system has a unique solution if f_1, f_2, f_3 and their first order partial derivatives are continuous not vanishing at the same point. Then, through each $M \in \boldsymbol{\Omega}$, exactly one vector line passes.

<u>Note</u>: Similarly, for a two-dimensional vector field we can get the differential equation $\dfrac{dx}{f_1(M)} = \dfrac{dy}{f_2(M)}$

<u>Example</u>

Find the vector line for the vector field $f(M) = x\,\boldsymbol{i} - y\,\boldsymbol{j} - 2z\,\boldsymbol{k}$ passing through the point M = [1, -1, 2].

Solution

We get the following system of differential equations

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{-2z} \quad \text{or} \quad \frac{dx}{x} = -\frac{dy}{y} \quad \frac{dy}{y} = \frac{dz}{2z}$$

this clearly has a solution

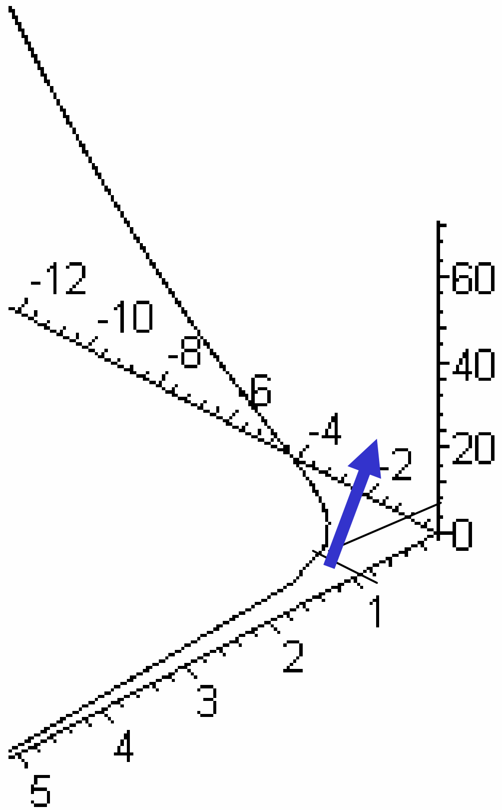
$xy = c_1, \quad y^2 = c_2 z$ or, using parametric equations,

$$x = \frac{c_1}{t},\ y = t,\ z = \frac{t^2}{c_2}, \quad t \neq -1$$

Since the resulting vector line should pass through M, we get

$$c_1 = -1, c_2 = \frac{1}{2}$$

$$x = -\frac{1}{t}, y = t, z = 2t^2, \quad t \neq -1$$



Example

Find the vector lines of a planar flow of fluid characterized by the field of velocities $\vec{f}(M) = xy\,\vec{i} + 2x(x-1)\,\vec{j}$

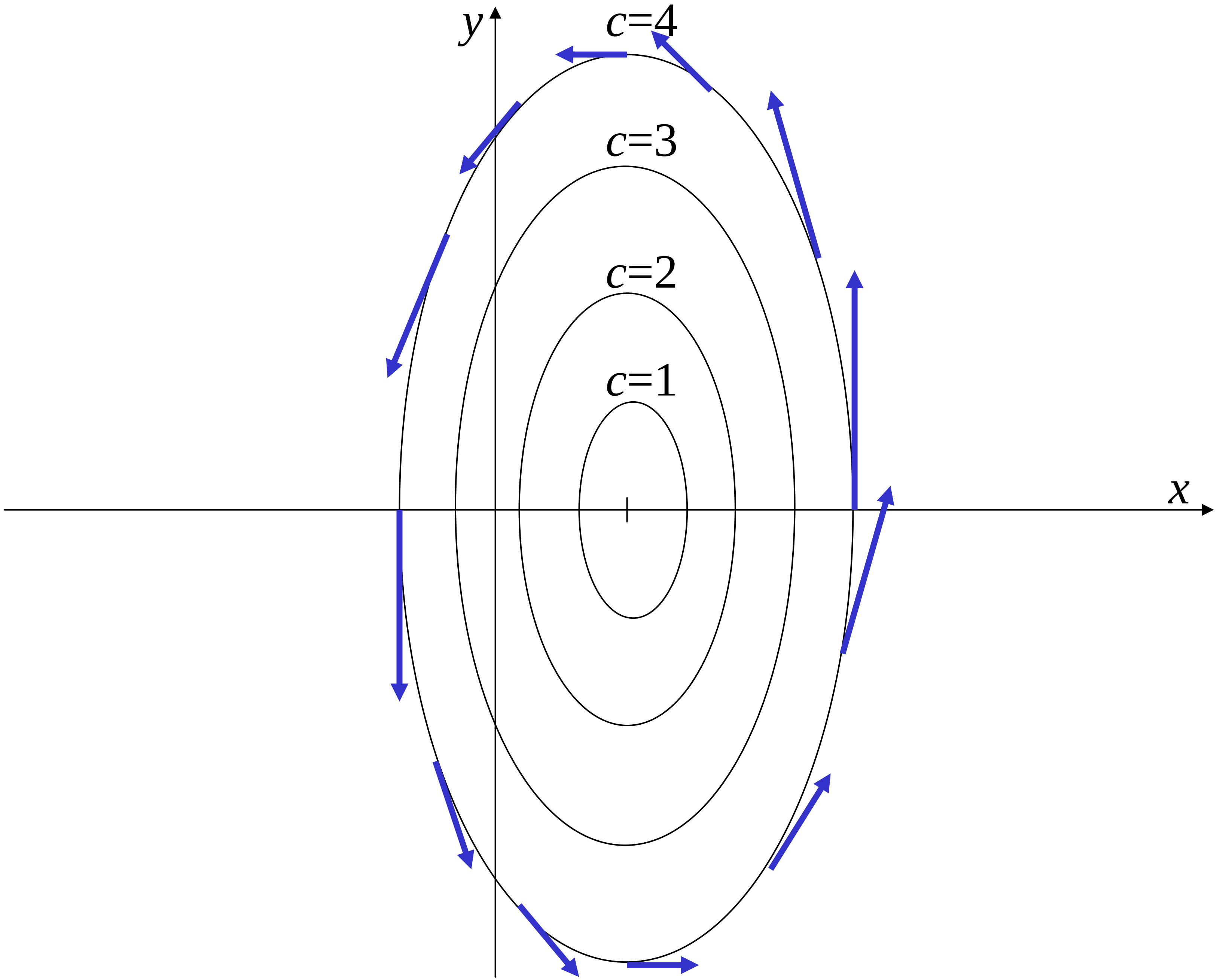
The differential equation defining the vector lines is

$$\frac{dx}{xy} = -\frac{dy}{2x(x-1)}$$

integrating this differential equation yields

$$2\int (x-1)\,dx = -\int y\,dy \Rightarrow x^2 - 2x = -\frac{1}{2}y^2 + c_1$$

and thus $\quad (x-1)^2 + \frac{y^2}{2} = c \; (c > 0)$



y
x
c=4
c=3
c=2
c=1

VECTOR FIELD

We will denote by $\vec{f}(M)$ a real vector function of a point M in an area A. If A is two dimensional, then $\vec{f}(M) = f_1(x, y)\vec{i} + f_2(x, y)\vec{j}$

and, if A is a 3-D region, then

$$\boxed{\vec{f}(M) = f_1(x, y)\vec{i} + f_2(x, y)\vec{j} + f_3(x, y)\vec{k}}$$

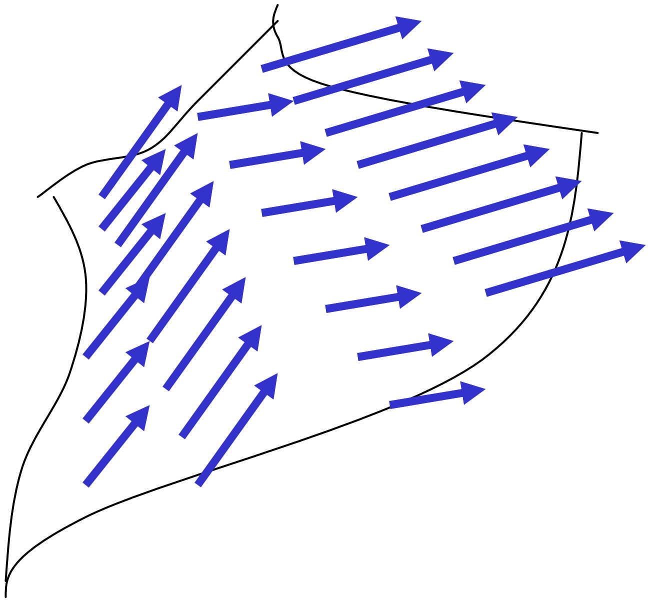
We will call $\vec{f}(M)$ a ***vector field*** in A.

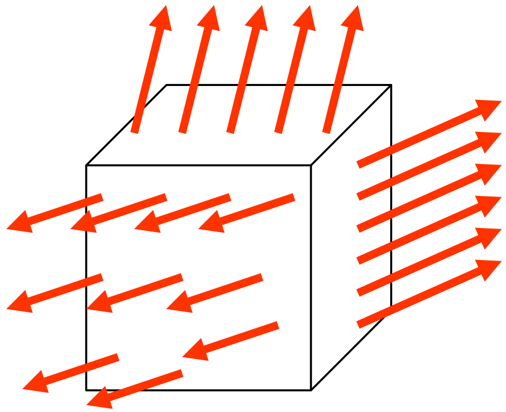
Flux through a surface

Let σ be a simple (closed) surface and $\vec{f}(M)$ a 3-D vector field. The surface integral

$$\iint_{\sigma} \vec{f}(M)\, d\vec{S} = \iint_{\sigma} f_1(x,y,z)\,dydz + f_2(x,y,z)\,dxdz + f_3(x,y,z)\,dxdy$$

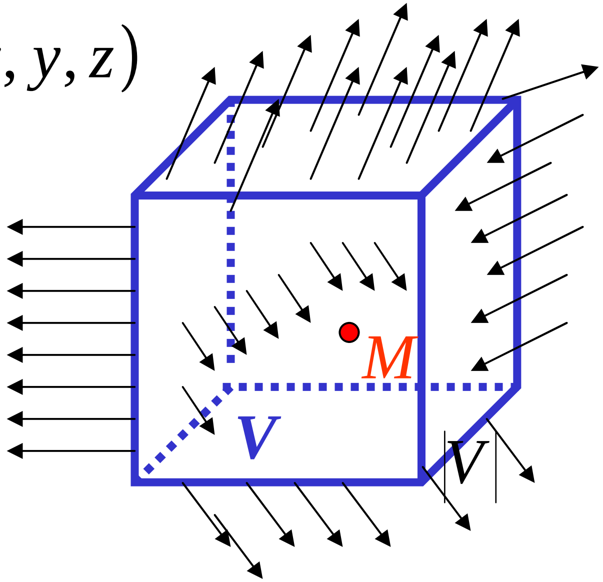
is called the flux of the vector field $\vec{f}(M)$ through surface σ.





Divergence of a vector field

vector field $\vec{f}(x, y, z)$



closed 3-D area V
with S as border

$$S = \partial \boldsymbol{V}$$

an internal point M

$$D = \frac{\iint\limits_{\partial V} \vec{f}(x, y, z) d\vec{S}}{|V|}$$

$|\boldsymbol{V}|$ is the volume of $\boldsymbol{V}$

D is the flux of $\vec{f}(x, y, z)$ through ∂V per unit volume

Let us shrink *V* to *M*, that is, the area *V* becomes a point and see what *D* does. If $\vec{f}(x, y, z)$ has continuous partial derivatives, the below limit exists, and we can write:

$$D(M) = \lim_{V \to M} D = \lim_{V \to M} \frac{\iint\limits_{\partial V} \vec{f}(x, y, z) d\vec{S}}{|V|}$$

If we view $\vec{f}(x, y, z)$ as the velocity of a fluid flow, *D*(*M*) represents the rate of fluid flow from *M*.

- for *D*(*M*) > 0, *M* is a source of fluid;
- for D(*M*) < 0, *M* is a sink.
- if *D*(*M*) = 0, then no fluid issues from *M*.

If we perform the above process for every M in the region in which the vector field $\vec{f}(x,y,z)$ is defined, we assign to the vector field $\vec{f}(x,y,z)$ a scalar field $D(x,y,z) = D(M)$.

This scalar field is called the **divergence** of $\vec{f}(x,y,z)$.

We use the following notation:

$$D(M) = \operatorname{div} \vec{f}(x,y,z) = \operatorname{div}\left(f_1(x,y,z)\vec{i} + f_2(x,y,z)\vec{j} + f_3(x,y,z)\vec{k} \right)$$

It can be proved that, in Cartesian coordinates, we have

$$\operatorname{div} \vec{f}(x,y,z) = \frac{\partial}{\partial x} f_1(x,y,z) + \frac{\partial}{\partial y} f_2(x,y,z) + \frac{\partial}{\partial z} f_3(x,y,z)$$

Or, using the Hamiltonian or nabla operator

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

we can write

$$\operatorname{div} \vec{f}(x,y,z) = \nabla \cdot \vec{f}(x,y,z)$$

Solenoidal vector field

If div div $\vec{f}(M) = 0$ at every M in a 3D region A, we say that the field $\vec{f}(x, y, z)$ is solenoidal in A.

If a vector field $\vec{f}(x, y, z)$ is solenoidal in a region A, then it has neither sources nor sinks in A.

<u>Gauss's - Ostrogradski's theorem</u>

Let us take a closed surface σ that contains a 3D region *V* where a vector field $\vec{f}(x, y, z)$ is defined and "add-up" the divergence of $\vec{f}(x, y, z)$ within *V*, that is, calculate the triple integral

$$\iiint_V \mathrm{div}\ \boldsymbol{f}(x, y, z)\,dx\,dy\,dz$$

The divergence being the rate of flow through points of 3D space, we see that this integral represents what flows through σ as the boundary of *V*. However this is exactly the flux of $\vec{f}(x, y, z)$ through σ

$$\iint_\sigma \boldsymbol{f}(x, y, z)\,d\boldsymbol{S}$$

If a vector function $\vec{f}(x, y, z)$ has continuous partial derivatives in a 3D region V bounded by a finite boundary σ, we can write

$$\oint\!\!\!\oint_{\sigma} \vec{f}(x, y, z)\, d\vec{S} = \iiint_{V} \operatorname{div} \vec{f}(x, y, z)\, dx\, dy\, dz$$

where the surface on the left-hand side of the equation is oriented so that the normals point outwards.

The most general form of this formula was first proved by Mikhail Vasilevich Ostrogradsky in 1828.

Example

Calculate the volume of an area V bounded by a closed surface σ given by the parametric equations

$$x = \varphi(u,v),\ y = \chi(u,v),\ z = \chi(u,v),\ [u,v] \in M$$

Let us define a vector field $\vec{f}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$

By Gauss's-Ostrogradsky's theorem, we have

$$\oiint_{\sigma} \vec{f}(x,y,z)d\vec{S} = \iiint_{V} \mathrm{div}\ \vec{f}(x,y,z)\,dx\,dy\,dz = \iiint_{V} 3\,dx\,dy\,dz$$

and thus

$$|V| = \frac{1}{3}\iint_{M} A\varphi'(u,v) + B\psi'(u,v) + C\chi'(u,v)\,du\,dv$$

where A, B, C are the coordinates of a normal to σ, that is,

$$A = \begin{vmatrix} \psi'_u & \chi'_u \\ \psi'_v & \chi'_v \end{vmatrix} \qquad B = \begin{vmatrix} \chi'_u & \varphi'_u \\ \chi'_v & \varphi'_v \end{vmatrix} \qquad C = \begin{vmatrix} \varphi'_u & \psi'_u \\ \varphi'_v & \psi'_v \end{vmatrix}$$

For example, if B is a ball of radius r, we have

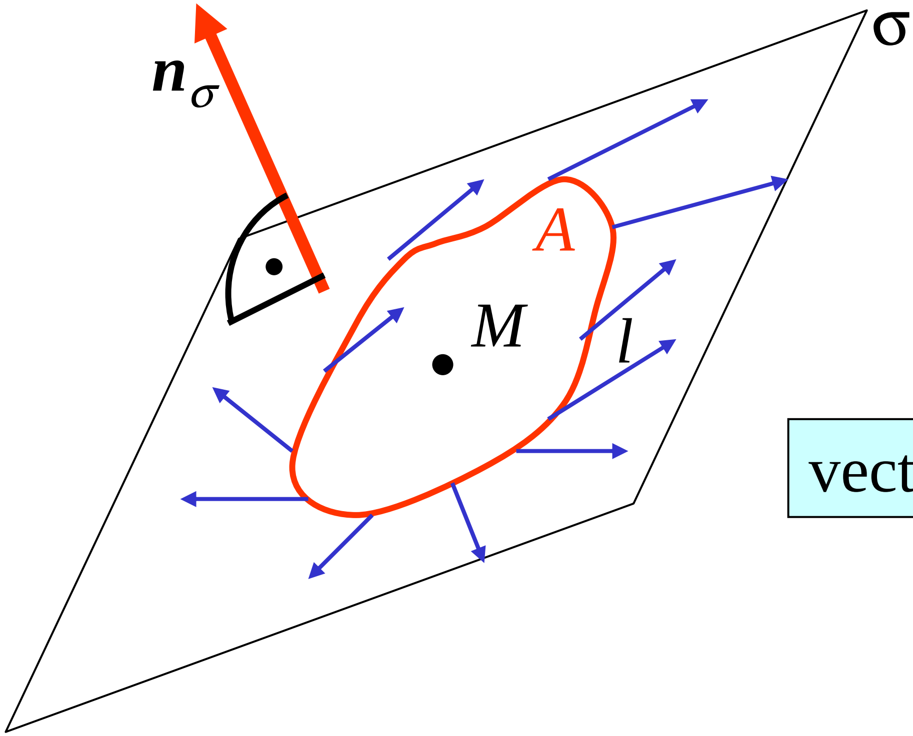
$$x = r\cos u \sin v,\ y = r\sin u \sin v,\ z = r\cos v,\ [u, v] \in [0, 2\pi] \times [0, \pi]$$

And so

$$|B| = \frac{1}{3}\iint_M -r^3\cos^2 u \sin^3 v - r^3\sin^2 u \sin^3 v - r^3 \sin v \cos^2 v \, du\, dv =$$

$$= \frac{1}{3}\iint_M -r^3 \sin v \, du\, dv = \frac{2\pi r^3}{3}\int_0^\pi \sin v \, dv = \frac{2\pi r^3}{3}[-\cos v]_0^\pi = \frac{4\pi r^3}{3}$$

The curl of a vector field



vector field $\vec{f}(x,y,z)$

For a plane σ determined by its unit normal $\vec{n}_\sigma$ containing a closed curve l with a fixed point M inside the area A bounded by l, define the quantity

$$C(\vec{n}_\sigma, l, M) = \frac{\oint\limits_l \vec{f}(x,y,z)\,d\vec{s}}{|A|}$$

$|A|$ is the surface area of A

If $\vec{f}(x, y, z)$ has continuous partial derivatives, we can calculate

$$C(n_\sigma, M) = \lim_{l \to M} C(n_\sigma, l, M)$$

and $C\,(\vec{n}_\sigma, M)$ does not depend on the choice of l and the way it shrinks to M. It is only determined by the point M and the direction of $\vec{n}_\sigma$

It can further be proved that there exists a universal vector $c(M)$ such that $c(M) \cdot \vec{n}_\sigma = C(\vec{n}_\sigma, M)$

for every normal $\vec{n}_\sigma$ determining the plane σ

Using the above method we can assign a vector **c**(M) to every point M in the 3D-region in which the vector field satisfies the assumptions (continuous partial derivatives). In other words, we have defined to the original vector field $\vec{f}(x, y, z)$ a new vector field $\vec{c}(x, y, z)$

This vector field is called the curl of $\vec{f}(x, y, z)$ and denoted by curl $\vec{f}(x, y, z)$ or rot $\vec{f}(x, y, z)$

$$\text{curl } f(x, y, z) = \left(\frac{\partial f_2}{\partial z} - \frac{\partial f_3}{\partial y}\right)\vec{i} + \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\vec{j} + \left(\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x}\right)\vec{k}$$

The last formula can be written using the following formal determinant

$$\operatorname{curl}\vec{f}(x,y,z)=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1(x,y,z) & f_2(x,y,z) & f_3(x,y,z) \end{vmatrix}$$

or

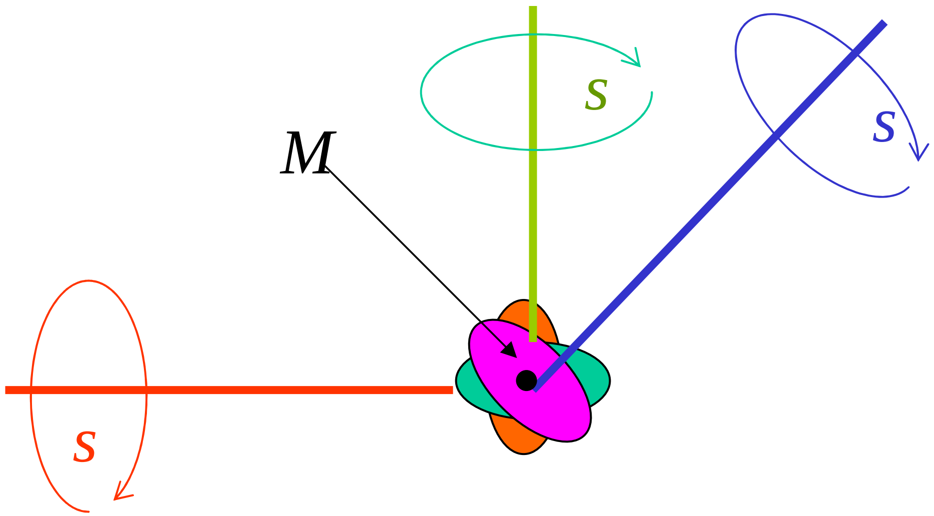
$$\operatorname{curl}\vec{f}(x,y,z)=\nabla\times\vec{f}(x,y,z)$$

Suppose that the vector field $\vec{f}(x, y, z)$ represents a flow of fluid.

Think of a very small turbine on a shaft positioned at a point M.

Through the fluid flow, the turbine will turn at a speed s.

Let us move the shaft changing its direction while leaving the turbine at the point M. In a certain direction $\vec{c}(M)$, the speed s of the turbine will reach its maximum. Then $\vec{c}(M)$ is clearly the curl of $\vec{f}(x, y, z)$ at M.



<u>**Irrotational vector field**</u>

If curl $\vec{f}(x, y, z) \equiv \vec{0}$ in a 3D-region A, we say that the field $\vec{f}(x, y, z)$ is **irrotational**.

Recall that $\vec{f}(x, y, z)$ has to be irrotational if line integrals of the vector field are to be independent of the line along which the integral is calculated being only functions of the initial and final points.

Every continuously differentiable vector function $\vec{f}(M)$ defined in a region A (subject to mild restrictions) can be expressed as a sum of two vector functions $\vec{g}(M)$ and $\vec{h}(M)$ such that $\vec{g}(M)$ is solenoidal and $\vec{h}(M)$ is irrotational.

The possibility of such decomposition greatly simplifies the study of many velocity and force fields occurring in physics.

Stokes formula

Let a vector field $\vec{f}(x, y, z)$ be defined in a 3D region A with continuous first partial derivatives. Let a simple 3D surface σ be given in A bounded by a closed regular curve l. Then the flux of the curl of $\vec{f}(x, y, z)$ through σ equals the line integral of . $.\vec{f}(x, y, z)$ along l. Formally

$$\iint_{\sigma} \operatorname{curl} \vec{f}(x, y, z) d\vec{S} = \oint_{l} \vec{f}(x, y, z) d\vec{s}$$

Example

Use the Stokes formula to calculate the second type line integral of the vector field $\vec{f}(x,y,z) = y\vec{i} + z\vec{j} + x\vec{k}$ along the circle C $x^2 + y^2 = r^2$.

We have curl $\vec{f} = -\vec{i} - \vec{j} - \vec{k}$

Consider the semi-sphere S:

$$x = r\cos u \sin v,\ y = r\sin u \sin v,\ z = r\cos v,\ [u,v] \in [0,2\pi] \times [0,\pi/2]$$

Since, clearly, C is the boundary of S, we can use the Stokes formula:

$$I = \oint_C y\,dx + z\,dy + x\,dz = \iint_S -dy\,dz - dx\,dz - dx\,dy =$$

$$= -\int_0^{2\pi} du \int_0^{\pi/2} \left(A + B + C\right) dv$$

where $A = -r^2 \cos u \sin^2 v,\ B = -r^2 \sin u \sin^2 v,\ C = -r^2 \sin v \cos v$

$$I = r^2 \int_0^{2\pi} du \int_0^{\pi/2} \left(\cos u \sin^2 v + \sin u \sin^2 v + \sin v \cos v\right) dv$$

$$I = r^2 \left\{ \int_0^{2\pi} \cos u\, du \int_0^{\pi/2} \sin^2 v\, dv + \int_0^{2\pi} \sin u\, du \int_0^{\pi/2} \sin^2 v\, dv + 2\pi \int_0^{\pi/2} \sin v \cos v\, dv \right\}$$

Since, clearly, $\int_0^{2\pi} \sin u \, du = \int_0^{2\pi} \cos u \, du = 0$, we have

$$I = 2\pi r^2 \int_0^{\pi/2} \sin v \cos v \, dv = \pi r^2 \int_0^{\pi/2} \sin 2v \, dv = \pi r^2 \frac{1}{2} \left[-\cos 2v \right]_0^{\pi/2} = \pi r^2$$