Ordinary differential equations - ODE

An n-th order ordinary differential equation (ODE_n) is an equation

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

where $F\left(x_1, x_2, \ldots, x_{n+1}\right) = 0$ is a known function in $n + 1$

variables and $y = y\,(x)$ is an unknown function with derivatives

$$y' = \frac{dy(x)}{dx}, y'' = \frac{d^2 y(x)}{dx^2}, \ldots y^{(n)} = \frac{d^n y(x)}{dx^n},$$

Examples

$$\sqrt{\ln y' x + \sin y} - \tan(xy) = 0 \qquad \text{ODE}_1$$

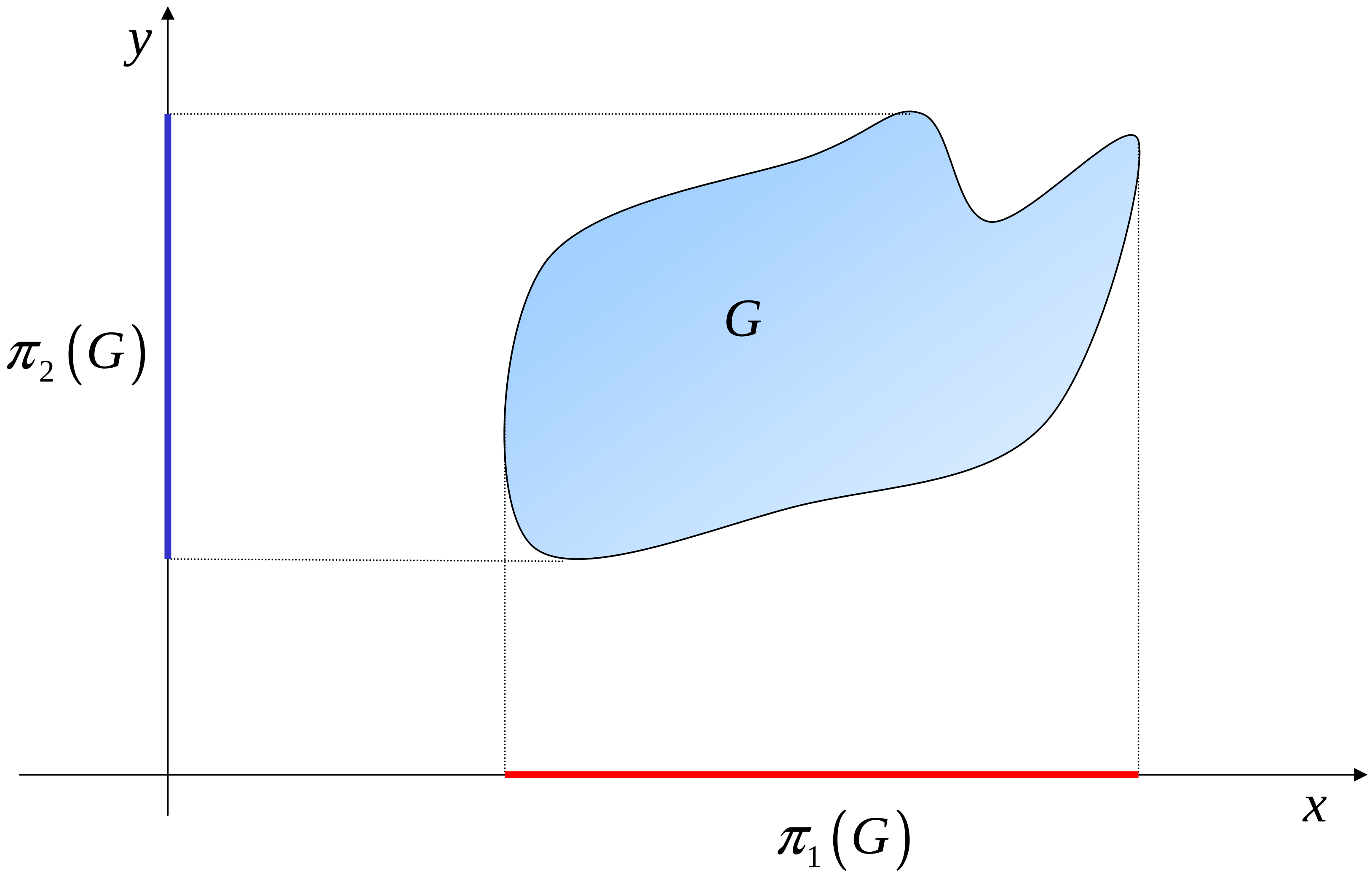
$$\sin y'' + \frac{\ln y'}{y^{(4)}} = 0 \qquad \text{ODE}_4$$

$$\sin xy''' - 2y'' + y' - \tan y + x = 0 \qquad \text{ODE}_3$$

$$y^{(5)} - 4y''' - 2y'' + y' - 5y + x = 0 \qquad \text{ODE}_5$$

Solution to a differential solution

Defining a solution to a general differential equation of order n in an exact way is rather a tricky business. However, since in most of the ordinary situations, this notion is quite intuitive, by way of an example, we will only define precisely what we understand under a solution to a first order differential solution.



$$\pi_1(G)=\{x \mid \exists y:[x,y]\in G\}$$

$$\pi_2(G)=\{y \mid \exists x:[x,y]\in G\}$$

Let $y' = f(x, y)$ be a differential equation where $f(x_1, x_2)$ is a continuous function on $G \subseteq E_2$.

We say that a function $u = u(x)$ is a solution to $y' = f(x, y)$ if

- $u(x)$ has a continuous derivative in some interval $J \subseteq \pi_1(G)$
- the entire graph of $u(x)$ lies in G, that is, $[x, y(x)] \in G, x \in J$
- when substituting u for y in and $\dfrac{du(x)}{dx}$ for y' in $y' = f(x, y)$

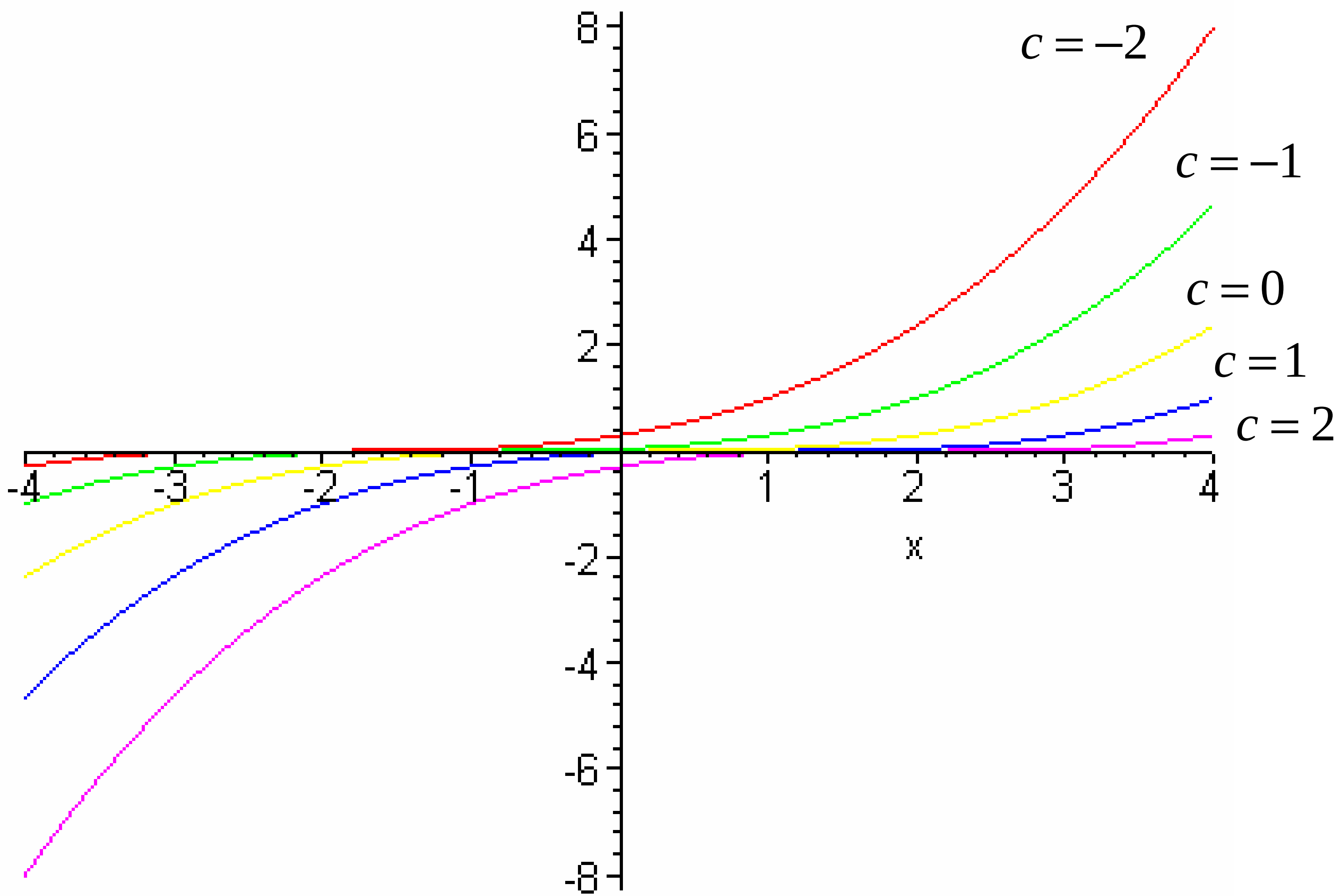
  the equation holds for every $x \in J$

Using another example, we shall now explain the difference between a **general** , **particular**, and **singular** solution to a differential equation:

Let us have a differential equation

$$y' = \sqrt[3]{y^2}$$

Since no other restrictions are given, G is E_2 and we can easily verify that $u = \frac{1}{27}(x+c)^3$ is a solution to $y' = \sqrt[3]{y^2}$ for any $c \in R$ and so is the function $u(x) \equiv 0$



8
6
4
2
-2
-4
-6
-8
-4
-3
-2
-1
1
2
3
4
x
$c = -2$
$c = -1$
$c = 0$
$c = 1$
$c = 2$

$u = \frac{1}{27}(x+c)^3$ is a general solution to $y' = \sqrt[3]{y^2}$

a **<u>general solution</u>** represents a set of solutions to a differential equation that only differ from each other by the values of a parameter (in a general case of several parameters) lying in a set of admissible values

$u = \frac{1}{27}(x+1)^3$ is a particular solution to $y' = \sqrt[3]{y^2}$

a **particular solution** is a solution to a differential equation that can be obtained from a general solution by choosing a value of the parameter (parameters)

$u(x) \equiv 0$ is a singular solution to $y' = \sqrt[3]{y^2}$

a **singular solution** to a differential equation is such a solution for which no value of the parameter in a general solution exists, that is, it cannot be obtained by substituting into a general solution.

Cauchy initial problem

We are to find a solution $y = y(x)$ to the differential equation

$$\boxed{y' = f(x, y)}$$

such that it satisfies an initial condition

$$\boxed{y(x_0) = y_0}$$

A solution to a Cauchy initial problem in the general case may not be easy to find. Sometimes it can even be proved that no "analytic" solution exists. A solution is called analytic if it can be expressed in terms of a finite set of "basic known functions" combining them using a finite number of arithmetic operations and embeddings.