Linear homogeneous ODE$_n$ with constant coefficients

(H) $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0$$

where $a_0, a_1, \ldots, a_n \in R,\ a_n \neq 0$

Let $y_1(x), y_2(x), \ldots, y_k(x),\quad k \leq n$ be solutions to (H), we say that they are independent if $c_1 y_1(x) + c_2 y_2(x) + \ldots + c_k y_k(x) \equiv 0$ implies

$$c_1 = c_2 = \cdots = c_k = 0$$

Let $y_1(x), y_2(x), \ldots, y_n(x)$ be *n* solutions to (H), the function

$$W[y_1, y_2, \ldots, y_n](x) = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y'_1(x) & y'_2(x) & \ldots & y'_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}$$

is called a Wronskian.

It has some interesting and useful properties

Let $y_1(x), y_2(x), \ldots, y_n(x)$ be n solutions to (H) on an interval [a,b].

Then they are independent exactly when there is an $x_0 \in [a,b]$ such that

$$W[y_1, y_2, \ldots, y_n](x_0) \neq 0$$

Linear non-homogeneous ODE$_n$ with constant coefficients

(N) $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = f(x)$$

where $a_0, a_1, \ldots, a_n \in R,\ a_n \neq 0$

associated homogeneous equation

(H) $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0$$

If $y(x)$ is a solution to (H) and $u(x)$ is a solution to (N) then $y(x) + u(x)$ is a solution to (N)

General solution to (H)

(H) $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0$$

Assume $\boxed{y(x) = e^{\lambda x}}$ substituting into (H) yields

$$a_n \lambda^n e^{\lambda x} + a_{n-1} \lambda^{n-1} e^{\lambda x} + a_{n-2} \lambda^{n-2} e^{\lambda x} + \cdots + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0$$

(E) $$a_n \lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \cdots + a_1 \lambda + a_0 = 0$$

If all the *n* roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the above equation are real and simple, we obtain the following general solution

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}$$

If λ is a root of (E) of multiplicity ***k***, then it can be proved that there are ***k*** independent solutions

$$\lambda$$

$$e^{\lambda x} \quad xe^{\lambda x} \quad x^2 e^{\lambda x} \cdots \quad x^{n-1} e^{\lambda x}$$

to the equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0$$

If $\boxed{a+ib}$ is a complex root of equation (E), then, since the coefficients are real, there must be also a root $\boxed{a-ib}$ and the following two independent functions

$$e^{ax}\cos bx, e^{ax}\sin bx$$

are solutions to (H). If $a+ib$ is a root of multiplicity *k*, then again we have the following 2*k* independent solutions of (H):

$$e^{ax}\cos bx, xe^{ax}\cos bx, x^2e^{ax}\cos bx, \ldots, x^{n-1}e^{ax}\cos bx,$$
$$e^{ax}\sin bx, xe^{ax}\sin bx, x^2e^{ax}\sin bx, \ldots, x^{n-1}e^{ax}\sin bx,$$

To find a particular solution to

(N) $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = f(x)$$

we can use the **variation-of-constants** method. Let

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

be a general solution to

(H) $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0$$

suppose that the constants c_1, ... , c_n become functions and let us substitute

$$y(x) = c_1(x) y_1(x) + c_2(x) y_2(x) + \cdots + c_n y_n(x)$$ into (N)

$$y' = c'_1 y_1 + \cdots + c'_n y_n + c_1 y'_1 + \cdots + c_n y'_n$$

if we put $c'_1 y_1 + \cdots + c'_n y_n = 0$ we have $y' = c_1 y'_1 + \cdots + c_n y'_n$

$$y'' = c'_1 y'_1 + \cdots + c'_n y'_n + c_1 y''_1 + \cdots + c_n y''_n$$

if we put $c'_1 y'_1 + \cdots + c'_n y'_n = 0$ we have $y'' = c_1 y''_1 + \cdots + c_n y''_n$

$$\vdots$$

$$y^{(n-1)} = c'_1 y_1^{(n-2)} + \cdots + c'_n y_n^{(n-2)} + c_1 y_1^{(n-1)} + \cdots + c_n y_n^{(n-1)}$$

if we put $c'_1 y_1^{(n-2)} + \cdots + c'_n y_n^{(n-2)} = 0$ we have

$y^{(n-1)} = c_1 y_1^{(n-1)} + \cdots + c_n y_n^{(n-1)}$ and finally

$$y^{(n)} = c'_1 y_1^{(n-1)} + \cdots + c'_n y_n^{(n-1)} + c_1 y_1^{(n)} + \cdots + c_n y_n^{(n)}$$

$$a_n\left(c_1 y_1^{(n)} + \cdots + c_n y_n^{(n)} + c'_1\, y_1^{(n-1)} + \cdots + c'_n\, y_n^{(n-1)}\right) +$$

$$a_{n-1}\left(c_1 y_1^{(n-1)} + \cdots + c_n y_n^{(n-1)}\right) +$$

$$a_1\left(c_1 y'_1 + \cdots + c_n y'_n\right) +$$

$$a_0\left(c_1 y_1 + \cdots + c_n y_n\right) = f(x)$$

0 ... 0

Thus, to find the functions $c'_1(x), c'_2(x), \ldots, c'_n(x)$, we have to solve the following system of *n* algebraic equations

$$
\begin{aligned}
c'_1 y_1 + \cdots + c'_n y_n &= 0 \\
c'_1 y'_1 + \cdots + c'_n y'_n &= 0 \\
&\vdots \\
c'_1 y_1^{(n-2)} + \cdots + c'_n y_n^{(n-2)} &= 0 \\
c'_1 y_1^{(n-1)} + \cdots + c'_n y_n^{(n-1)} &= f(x)
\end{aligned}
\qquad \text{(S)}
$$

This system has a solution since the determinant of the system is the Wronskian $W[y_1, y_2, \ldots, y_n](x_0)$ which is non-zero due to the fact that $y_1, y_2, \ldots, y_n$ are independent.

If $u_1(x), u_2(x), \ldots, u_n(x)$ are the functions solving the system (S), we get

$$c_1(x) = \int u_1(x)\,dx$$

$$c_2(x) = \int u_2(x)\,dx$$

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$c_n(x) = \int u_n(x)\,dx$ and a general solution to (H)

$$\boxed{y(x) = \int u_1(x)\,dx \cdot y_1(x) + \int u_2(x)\,dx \cdot y_2(x) + \cdots + \int u_n(x)\,dx \cdot y_n(x)}$$

Example

$$y''' + 2y'' + y' = x$$

$$y''' + 2y'' + y' = 0 \qquad \lambda^3 + \lambda^2 + \lambda = 0, \quad \lambda_1 = 0, \lambda_{2,3} = -1$$

$$y_1 = 1, y_2 = e^{-x}, y_3 = xe^{-x} \qquad y = c_1 + c_2 e^{-x} + c_3 xe^{-x} + u$$

where u is a particular solution to the original non-homogeneous solution

$$y''' + 2y'' + y' = x$$

Let $u = c_1(x) + c_2(x) e^{-x} + c_3(x) xe^{-x}$

We have to solve the following system of algebraic equations

$$\begin{array}{lllllll} c'_1 & + & c'_2 e^{-x} & + & c'_3 x e^{-x} & = & 0 \\ & & -c'_2 e^{-x} & + & c'_3 (1-x) e^{-x} & = & 0 \\ & & c'_2 e^{-x} & + & c'_3 (x-2) e^{-x} & = & x \end{array}$$

If we denote by A the system matrix

$$\begin{pmatrix} 1 & e^{-x} & x e^{-x} \\ 0 & -e^{-x} & (1-x) e^{-x} \\ 0 & e^{-x} & (x-2) e^{-x} \end{pmatrix}$$

then, clearly, $\left(c'_1, c'_2, c'_3\right)^T = A^{-1} (0, 0, x)^T$

Calculating A^{-1}, we get

$$\begin{pmatrix} c'_1 \\ c'_2 \\ c'_2 \end{pmatrix} = e^{2x} \begin{pmatrix} e^{-2x} & -e^{-2x} & e^{-2x} \\ 0 & (x-2)e^{-x} & (x-1)e^{-x} \\ 0 & -e^{-x} & -e^{-x} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$$

so that $c'_1 = x, c'_2 = e^x\left(x^2 - x\right), c'_3 = -xe^x$ and

$$c_1 = \int x\,dx, c_2 = \int e^x\left(x^2 - x\right)dx, c_3 = \int -xe^x dx$$

Calculating the integrals, we can now write a general solution of the original non-homogeneous differential equation $y''' + 2y'' + y' = x$

$$y = c_1 + c_2 e^{-x} + c_3 x e^{-x} + \frac{x^2}{2} - 2x + 3$$

Note that we could also find a particular solution u to the original non-homogeneous equation by assuming it in the form

$$u = ax^2 + bx$$

Assuming just $u = ax + b$ would not do since 0 is a solution to the equation

$$\lambda^3 + \lambda^2 + \lambda = 0$$

Substituting *u*, *u*’, *u*’’, and *u*’’’ into the non-homogeneous equation yields

$$0 + 2 \cdot 2a + 2ax + b = x$$

which means that $a = \dfrac{1}{2}$ and $b = -2$ and we have

$$u = \frac{x^2}{2} - 2x$$

which is another particular solution to $y''' + 2y'' + y' = x$

This method can be generalized for some special types of the right-hand sides

If

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = P_k(x)$$

where $P_k(x)$ is a polynomial of degree k

we can assume a solution of the form $y = Q_k(x)$

where $Q_k(x)$ is a polynomial of degree k

or

$y = x^{m-1} Q_k(x)$ if 0 is a root of (E) of multiplicity m

If

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = e^{\alpha x}\left(P_k(x)\cos(\beta x) + Q_l(x)\sin(\beta x)\right)$$

where $P_k(x)$ is a polynomial of degree k, $Q_l(x)$ is a polynomial of degree l,

we can assume a solution of the form $e^{\alpha x}\left(R_m(x)\cos(\beta x) + S_m(x)\sin(\beta x)\right)$

where $R_m(x)$ and $S_m(x)$ are polynomials of degree m with $m = \max\{k, l\}$

or

$$e^{\alpha x}\left(x^{k-1}R_m(x)\cos(\beta x) + x^{k-1}S_m(x)\sin(\beta x)\right)$$

if $\alpha + i\beta$ is a root of (E) of multiplicity m