We will be looking for a solution $\left(y_1(x), y_2(x), \ldots, y_n(x)\right)^T$

to the system of linear differential equations with constant

coefficients in a cannonical form

$$
\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + f_1(x) \\
y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + f_2(x) \\
&\vdots \\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + f_n(x)
\end{aligned}
$$

Satisfying the initial condition

$$\left(y_1(x_0), y_2(x_0), \ldots, y_n(x_0)\right)^T = \left(b_1, b_2, \ldots, b_n\right)^T$$

As the first step, we will find a general solution to the associated homogeneous system

$$
\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\
y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\
&\vdots \\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n
\end{aligned}
$$

It can be proved that such a system has n independent vector solutions, which can be displayed as columns in the matrix

$$
Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}
$$

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}$$

is called the fundamental solution matrix to the system

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned}$$

Denoting $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$

The system

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned}$$

can be written as $\boxed{\left(y_1', y_2', \dots, y_n'\right)^T = A\left(y_1, y_2, \dots, y_2\right)^T}$

The derivative of the fundamental solution matrix

$$Y' = \begin{pmatrix} y'_{12} & y'_{12} & \cdots & y'_{1n} \\ y'_{21} & y'_{22} & \cdots & y'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y'_{n1} & y'_{n2} & \cdots & y'_{nn} \end{pmatrix}$$

satisfies the matrix equation

$$Y' = AY$$

Let us examine the matrix function

$$e^{xA} = E + \frac{x}{1!}A + \frac{x^2}{2!}A^2 + \frac{x^3}{3!}A^3 + \cdots$$

We have

$$\frac{de^{xA}}{dx} = A + \frac{x}{1!}A^2 + \frac{x^2}{2!}A^3 + \cdots = Ae^{xA}$$

so that e^{xA} satisfies the matrix differential equation $Y' = AY$

The matrix e^{xA} is a fundamental solution matrix to the system

$$
\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\
y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\
&\vdots \\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n
\end{aligned}
$$

It can be proved that any solution to the above system can be written as

$$
\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = e^{xA} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}
$$

where

$c_1, c_2, \ldots, c_n$

are constants

The general solution to the system

$$
\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + f_1(x) \\
y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + f_2(x) \\
&\vdots \\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + f_n(x)
\end{aligned}
$$

can be written as

$$
\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = e^{xA} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \\ \vdots \\ \overline{y_n} \end{pmatrix}
$$

where $\left(\overline{y_1}, \overline{y_2}, \ldots, \overline{y_n}\right)^T$ is any solution to

We need to find a particular solution to the system

$$
\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + f_1(x) \\
y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + f_2(x) \\
&\vdots \\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + f_n(x)
\end{aligned}
$$

To this end we will again use the variation-of-constants method

Let Y be a fundamental matrix solution to the system

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned}$$

and $\begin{pmatrix} c_1(x) \\ c_2(x) \\ \vdots \\ c_n(x) \end{pmatrix}$ a vector of differentiable functions

Substituting

$$\begin{pmatrix} \overline{y_1} \\ \overline{y_2} \\ \vdots \\ \overline{y_n} \end{pmatrix} = Y \begin{pmatrix} c_1(x) \\ c_2(x) \\ \vdots \\ c_n(x) \end{pmatrix}$$

into

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + f_1(x) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + f_2(x) \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + f_n(x) \end{aligned}$$

yields $$Y'\begin{pmatrix} c_1(x) \\ c_2(x) \\ \vdots \\ c_n(x) \end{pmatrix} + Y\begin{pmatrix} c_1'(x) \\ c_2'(x) \\ \vdots \\ c_n'(x) \end{pmatrix} = AY\begin{pmatrix} c_1(x) \\ c_2(x) \\ \vdots \\ c_n(x) \end{pmatrix} + \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

but, since Y is a solution to the associated homogeneous system,

we have $Y' = AY$ and thus

$$Y'\begin{pmatrix} c_1(x) \\ c_2(x) \\ \vdots \\ c_n(x) \end{pmatrix} = AY\begin{pmatrix} c_1(x) \\ c_2(x) \\ \vdots \\ c_n(x) \end{pmatrix}$$

This leaves us with the system

$$Y\begin{pmatrix} c_1'(x) \\ c_2'(x) \\ \vdots \\ c_n'(x) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

This is basically a system of linear algebraic equations with

$$\begin{pmatrix} c_1'(x) \\ c_2'(x) \\ \vdots \\ c_n'(x) \end{pmatrix}$$

as the vector of unknowns

$$Y$$

as the system matrix

$$\begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

as the right-hand-side column

If $\begin{pmatrix} h_1(x) \\ h_2(x) \\ \vdots \\ h_n(x) \end{pmatrix}$ is a solution to $Y\begin{pmatrix} c_1'(x) \\ c_2'(x) \\ \vdots \\ c_n'(x) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$

$$\begin{pmatrix} \overline{y_1} \\ \overline{y_2} \\ \vdots \\ \overline{y_n} \end{pmatrix} = Y\begin{pmatrix} \int h_1(x)\,dx \\ \int h_2(x)\,dx \\ \vdots \\ \int h_n(x)\,dx \end{pmatrix}$$

is the particular solution we are looking for

If, in addition, the solution has to satisfy a set of initial conditions

$\left(y_1^0, y_2^0, \ldots, y_n^0\right)$ at a point x_0, we have to find the coefficients

c_1, c_2, ..., c_n in the general solution

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = e^{xA} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \\ \vdots \\ \overline{y_n} \end{pmatrix}$$

This can be done by solving the system of linear algebraic equations

$$\begin{pmatrix} y_1^0 \\ y_2^0 \\ \vdots \\ y_n^0 \end{pmatrix} = e^{x_0 A} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} \overline{y_1}(x_0) \\ \overline{y_2}(x_0) \\ \vdots \\ \overline{y_n}(x_0) \end{pmatrix}$$