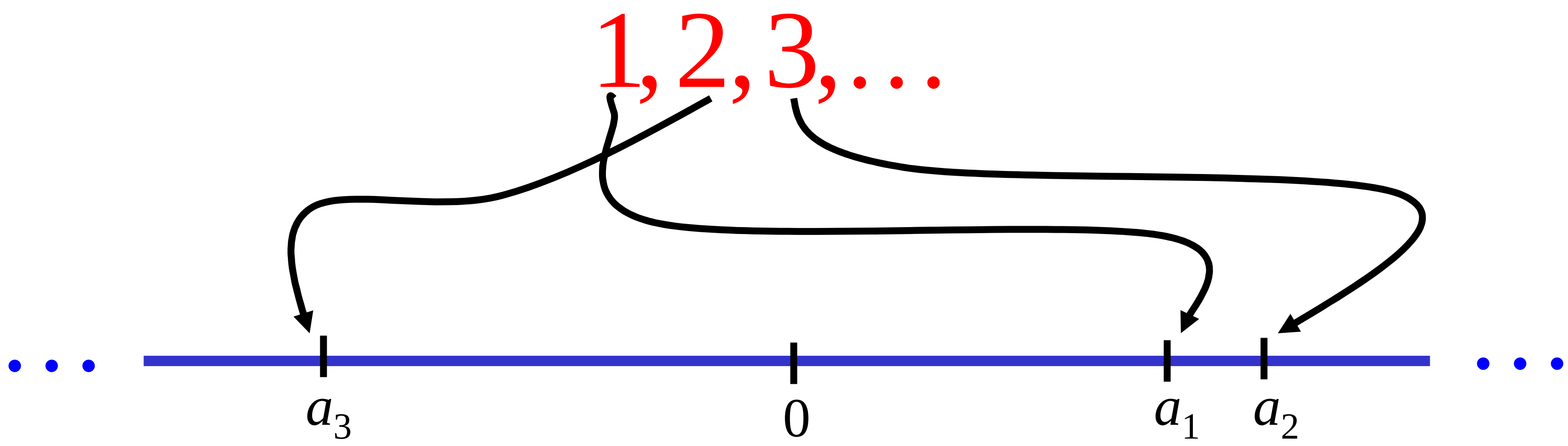
An (infinite) sequence of real numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

may be viewed as a mapping of natural numbers into real numbers



sometimes denoted $\{a_n\}_{n=1}^{\infty}$ or just $\{a_n\}$

For a given sequence $s = \{a_n\}_{n=1}^{\infty}$, we say that almost all of its terms have a property P if an index N exists such that a_n has the property P whenever $n > N$.

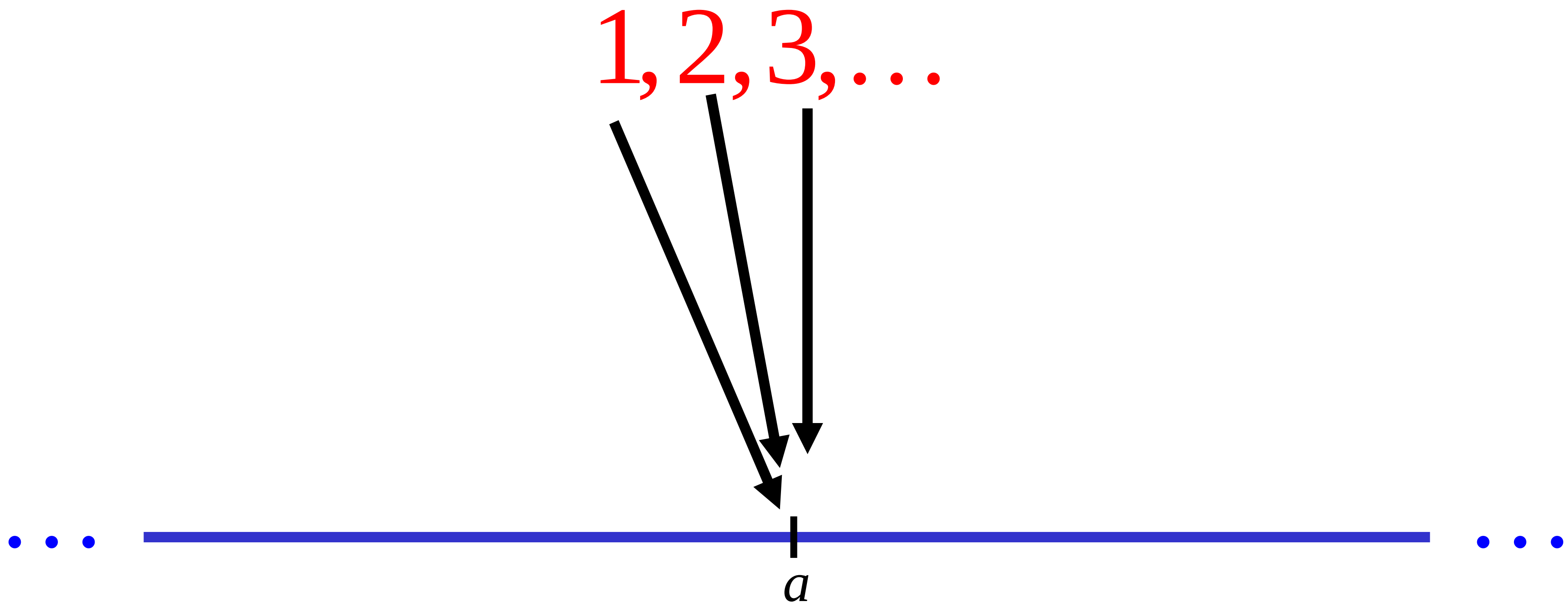
For example allmost all of the terms of the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$$

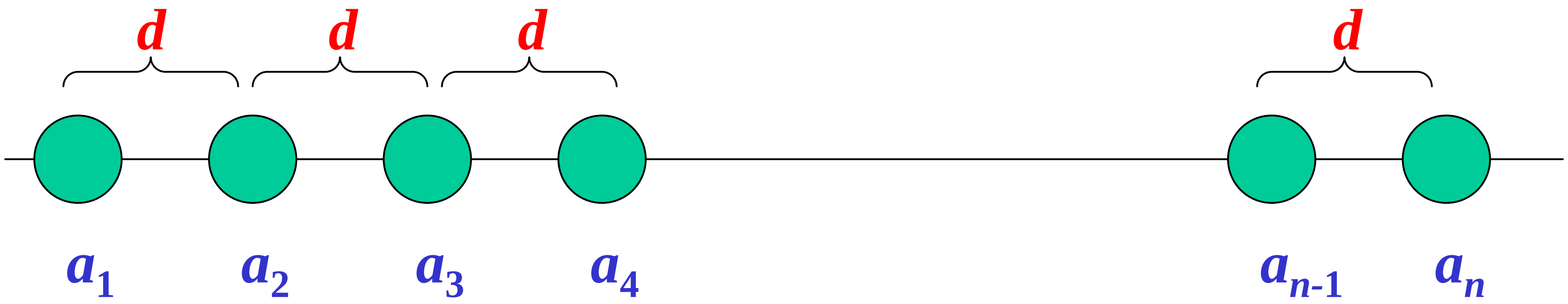
are less than 0.000 000 000 000 000 000 000 000 000 000 001

Constant sequence

$$a, a, a, \ldots, a, \ldots$$



Arithmetic sequence



$$a_{n+1} = a_n + d$$

$$a_1 = a$$

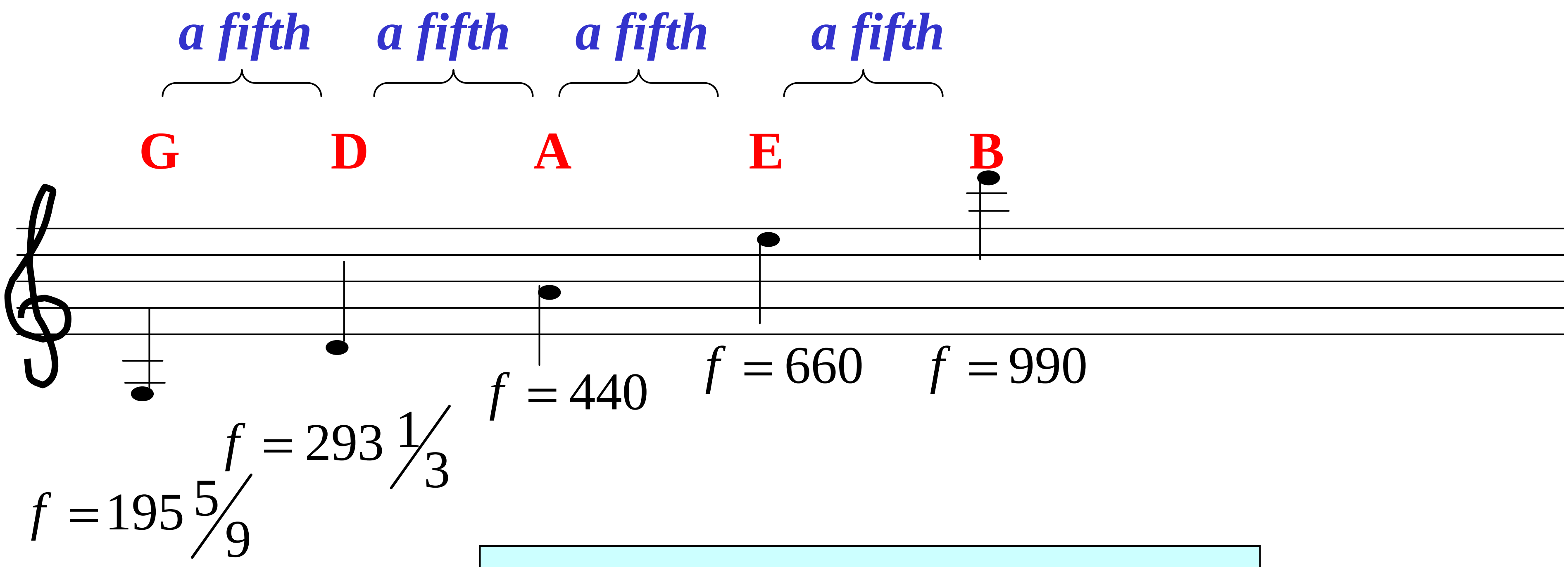
$$a_{n+1} = a + nd$$

Geometric sequence

$$a_{n+1} = q a_n$$

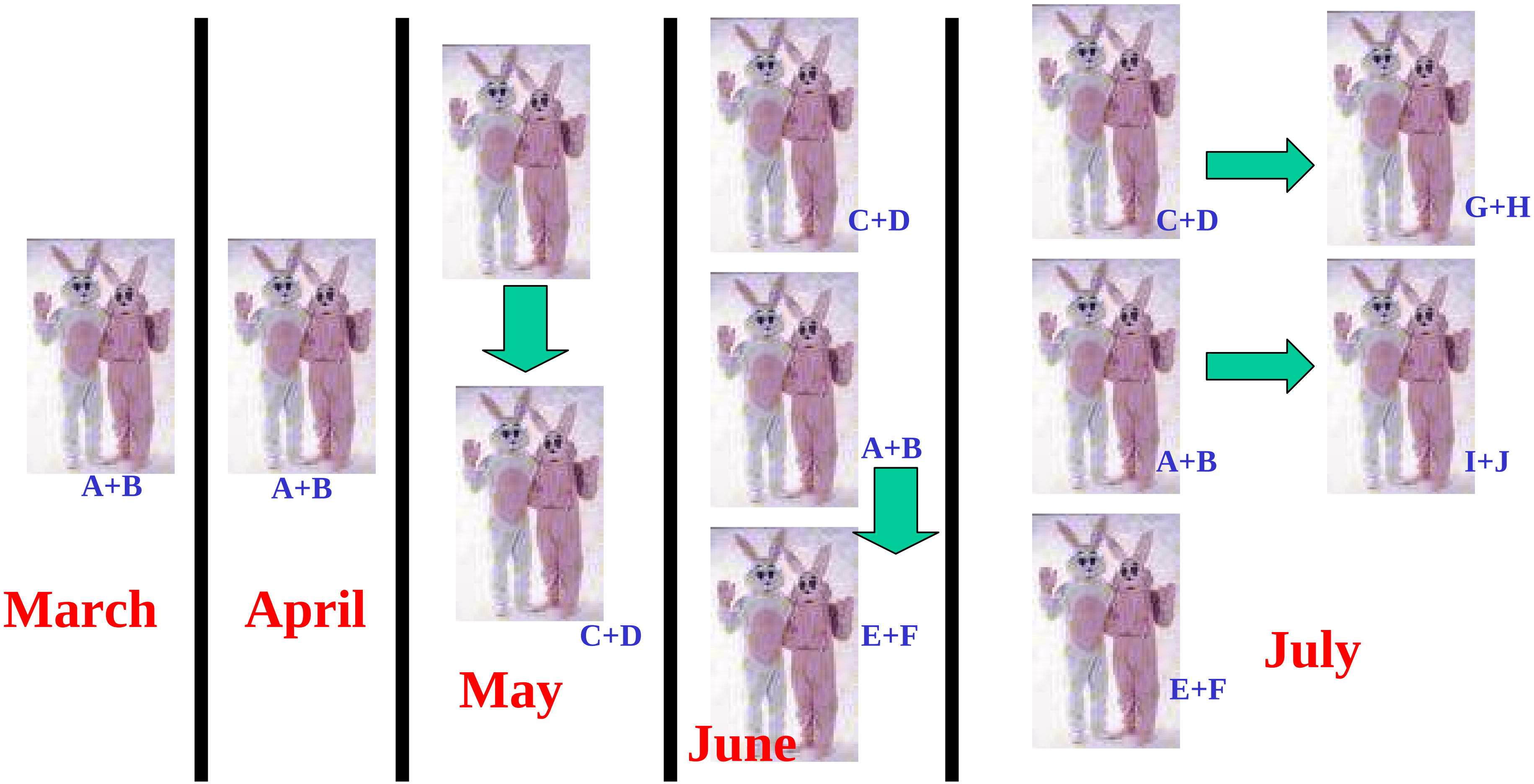
$$a_1 = a$$

$$a_{n+1} = q^n a$$



the quotient in a fifth is *q* = 3:2

In 1202 the mathematician **Leonardo of Pisa**, also called **Fibonacci**, published an influential treatise, *Liber abaci*. It contained the following recreational problem: "How many pairs of rabbits can be produced from a single pair in one year if it is assumed that every month each pair begets a new pair which from the second month becomes productive?"



The preceding example has shown that a sequence may also be defined using recurrence formulas, that is, defining the first few terms of the sequence and then giving a formula expressing the *n*-th sequence term through some of the preceeding terms. The Fibonacci “rabit-pair” sequence might be defined as follows:

$$a_1 = 1, a_2 = 1$$
$$a_{n+1} = a_n + a_{n-1}$$

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots$$

Explicit formulas may also be found equivalent to the recursive definition

$$a_n = \lambda^n$$

$$\lambda^{n+1} = \lambda^n + \lambda^{n-1}$$

$$\lambda^2 = \lambda + 1$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

$$a_n = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^n + \beta\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right) + \beta\left(\frac{1-\sqrt{5}}{2}\right)$$

$$1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^2 + \beta\left(\frac{1-\sqrt{5}}{2}\right)^2$$

$$\alpha = \frac{\sqrt{5}}{20} - \frac{1}{4}, \beta = \frac{-\sqrt{5}}{20} - \frac{1}{4}$$

$$\boxed{a_n = \left(\frac{\sqrt{5}}{20} - \frac{1}{4}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{-\sqrt{5}}{20} - \frac{1}{4}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

$a_1 < a_2 < a_3 < \cdots$ (strictly) increasing sequence

$a_1 \le a_2 \le a_3 \le \cdots$ non-decreasing sequence

$a_1 > a_2 > a_3 > \cdots$ (strictly) decreasing sequence

$a_1 \ge a_2 \ge a_3 \ge \cdots$ non-increasing sequence

$a_1, a_2, a_3, \ldots \quad a_i, \ldots \leq a$ sequence bounded above with *a* as an upper bound

$a_1, a_2, a_3, \ldots \quad a_i, \ldots \geq b$ sequence bounded below with *b* as a lower bound

$a_1, a_2, a_3, \ldots \quad b \leq a_i \leq a$ bounded sequence

<u>Subsequence</u>

Let $\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots$ be a sequence and let $i_1, i_2, i_3, \ldots$ be an increasing sequence of natural numbers (indices). Then we have sequence $\{a_{i_n}\}_{n=1}^{\infty} = a_{i_1}, a_{i_2}, a_{i_3}, \ldots$, which is called a subsequence of $\{a_n\}_{n=1}^{\infty}$. Sometimes $\{a_{i_n}\}_{n=1}^{\infty}$ is said to be selected from $\{a_n\}_{n=1}^{\infty}$ by discarding some of its terms while still leaving an infinite number of them.

$$\cancel{a_1} \quad a_2 \quad a_3 \quad \cancel{a_4} \quad \cancel{a_5} \quad a_6 \quad \cancel{a_7} \quad \cancel{a_8} \quad a_9 \quad \cdots\cdots$$

$$i_1 = 2, i_2 = 3, i_3 = 6, i_4 = 9, \ldots$$

A sequence $\{a_n\}_{n=1}^{\infty}$ is bounded iff the sequence $\{a_n\}_{n=k}^{\infty}$ is bounded for an arbitrary positive integer k

$$a_1, a_2, \ldots, a_{k-1}, a_k, a_{k+1}, \cdots$$

always exist

$$\bar{b} = \min\{a_1, a_2, \ldots, a_{k-1}\}$$

$$\bar{a} = \max\{a_1, a_2, \ldots, a_{k-1}\}$$

$$b \le a_j \le a,\ j = k, k+1, \ldots$$

$$\min\left\{\bar{b}, b\right\} \le a_i \le \max\left\{\bar{a}, a\right\}, i = 1, 2, \ldots$$

The limit of a sequence

We say that a sequence $\{a_n\}_{n=1}^{\infty}$ has a limit *A*, formally

$$\lim_{n\to\infty} a_n = A$$

if, for every $\varepsilon > 0$, there exists *N* such that

$$|A - a_n| < \varepsilon$$

for every $n > N$

Examples

$r, r, r, \ldots$ $\qquad \lim_{n\to\infty} r = r$

$\left\{\dfrac{1}{2^{n-1}}\right\} = 1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \ldots$ $\qquad \lim_{n\to\infty} \dfrac{1}{2^{n-1}} = 0$

$\left\{(-1)^{n-1}\right\} = 1, -1, -1, 1, -1, 1, \ldots$ no limit exists

$\{2n-1\} = 1, 3, 5, \ldots\ldots$ strictly speaking, no limit exists, but in this case, sometimes ∞ is taken to be one

Every sequence has at most one limit

Rules for calculating limits of sequences:

- $$\lim_{n\to\infty}\left(a_n \pm b_n\right) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$
- $$\lim_{n\to\infty}\left(a_n \cdot b_n\right) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$
- $$\lim_{n\to\infty}\left(a_n / b_n\right) = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n \text{ if } \lim_{n\to\infty} b_n \neq 0$$

provided that both limits on the right-hand sides exist

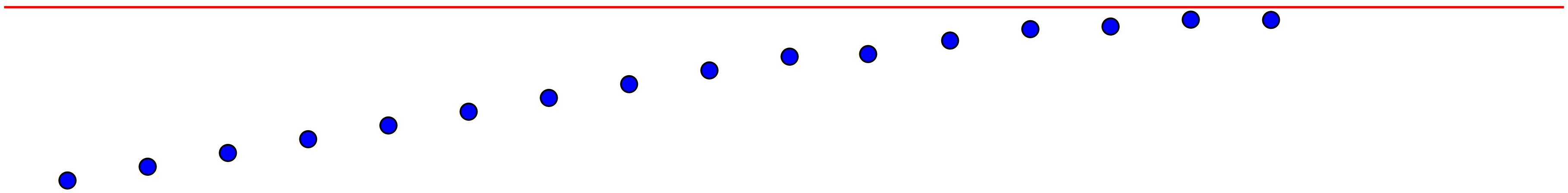
$$\lim_{n\to\infty}\frac{1}{a_n}=0 \text{ if } \lim_{n\to\infty}a_n=\infty$$

If $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=A$ and $a_n\le c_n\le b_n$ for every $n>N$, then

$$\lim_{n\to\infty}c_n=A$$

"*squeezing rule*"

If a sequence $\{a_n\}$ is bounded above and non-decreasing, it has a limit



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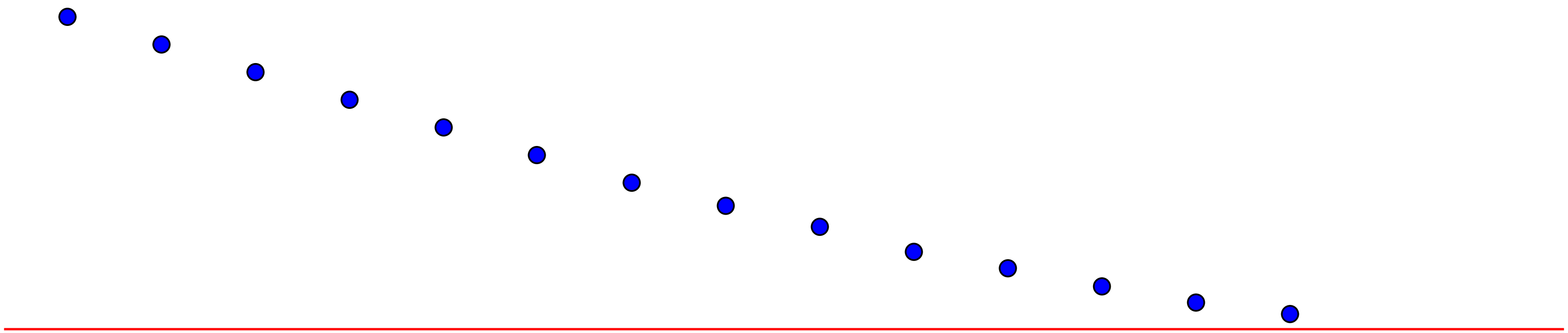

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and let

$$\lim_{n\to\infty} a_n = A$$

Let $\{a_{i_n}\}_{n=1}^{\infty}$ be an arbitrary subsequence of $\{a_n\}_{n=1}^{\infty}$, then

- $\{a_{i_n}\}_{n=1}^{\infty}$ has a limit and
- $\lim_{n\to\infty} a_{i_n} = A$

Point of condensation

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. We say that a is a point of condensation of $\{a_n\}_{n=1}^{\infty}$ if, for every $\varepsilon > 0$, we have $|a - a_n| < \varepsilon$ for an infinite number of indices

$\Updownarrow$

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. We say that a is a point of condensation of $\{a_n\}_{n=1}^{\infty}$ if, for every $\varepsilon > 0$, and for every index N, there exists an index $n > N$ such that $|a - a_n| < \varepsilon$

Example

$$1,1,-1,\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{2}{3},\frac{1}{3},\frac{-2}{3},\frac{3}{4},\frac{1}{4},\frac{-3}{4},\ldots,1-\frac{1}{n},\frac{1}{n},-1+\frac{1}{n},\ldots$$

It is easy to see that the points of condensation of the above sequence are the numbers

$$1,0,-1$$

- Every bounded sequence has at least one point of condensation.
- From every bounded sequence, we can select a subsequence that is convergent.
- The set of all the points of condensation of a sequence has the least and the greatest element.

The greatest point u of condensation of a sequence $\{a_n\}_{n=1}^{\infty}$ is called an upper limit of $\{a_n\}_{n=1}^{\infty}$ (or a limes superior).

$$\boxed{u = \limsup_{n\to\infty} a_n}$$

The lowest point l of condensation of a sequence $\{a_n\}_{n=1}^{\infty}$ is called a lower limit of $\{a_n\}_{n=1}^{\infty}$ (or a limes inferior).

$$\boxed{l = \liminf_{n\to\infty} a_n}$$

Example

$$a_n = \frac{\left(1+(-1)^n\right)(n+1)}{n}$$

$$0, \frac{2}{2}, 0, \frac{10}{4}, 0, \frac{14}{6}, 0, \frac{18}{8}, \ldots$$

$$\liminf_{n\to\infty} \frac{\left(1+(-1)^n\right)(n+1)}{n} = 0$$

$$\limsup_{n\to\infty} \frac{\left(1+(-1)^n\right)(n+1)}{n} = 2$$