An important example

The sequence $a_n = \left(1+\dfrac{1}{n}\right)^n$ is increasing while the sequence $b_n = \left(1+\dfrac{1}{n}\right)^{n+1}$ is decreasing and both have the same limit,

which we will denote by *e*.

Note: The limit *e* can be numerically calculated

$$e = 2{,}718\,281\,828\,459\,045\ldots$$

It is an irrational number sometimes referred to as Euler's constant. It is a very important number if not the most important one in mathematics being the base of natural logarithms

Proof

The inequality $\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2}$, which proves that b_n is decreasing is equivalent to the following inequalities

$$\left(\frac{n+1}{n}\right)^{n+1} > \left(\frac{n+2}{n+1}\right)^{n+2} \Leftrightarrow \left(\frac{n+1}{n}\right)^{n+2} > \left(\frac{n+2}{n+1}\right)^{n+2} \frac{n+1}{n} \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{(n+1)^2}{n(n+2)}\right)^{n+2} > 1+\frac{1}{n} \Leftrightarrow \left(1+\frac{1}{n(n+2)}\right)^{n+2} > 1+\frac{1}{n}$$

The last equivalence is true due to the fact that

$$(n+1)^2 = n(n+2)+1$$

Now it is not difficult to verify that, for $h > 0$ and any integer $k > 1$, we have $(1+h)^k > 1+kh$. Hence

$$\left(1+\frac{1}{n(n+2)}\right)^{n+2} > 1+\frac{n+2}{n(n+2)} = 1+\frac{1}{n}$$

Since $\left(1+\frac{1}{n}\right)^{n+1} > 0$, $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}_{n=1}^{\infty}$ is bounded below and so

$\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = e$ Then we can write:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n+1}}{1+\frac{1}{n}} = \frac{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)} = \frac{e}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)} = e$$

The last thing we have to prove is that $a_n < a_{n+1}$. Using the binomial theorem:

$$\left(x+y\right)^n = x^n + \frac{n!}{1!\left(n-1\right)!}x^{n-1}y + \frac{n!}{2!\left(n-2\right)!}x^{n-2}y^2 + \cdots + \frac{n!}{\left(n-1\right)!1!}xy^{n-1} + y^n$$

we can write

$$a_n = 1 + \frac{n!}{1!\left(n-1\right)!}\frac{1}{n} + \frac{n!}{2!\left(n-2\right)!}\frac{1}{n^2} + \cdots + \frac{n!}{\left(n-1\right)!1!}\frac{1}{n^{n-1}} + \frac{1}{n^n}$$

$$(1) \quad a_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots +$$

$$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

Substituting $n + 1$ for n in (1), we get

$$(2)\quad a_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+\cdots+$$

$$+\frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)+$$

$$+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

Matching the corresponding terms in (1) and (2) we see that, starting from the third, each term in (1) is less than its counterpart in (2) with (2) having one more positive term. Thus we conclude that that $a_n < a_{n+1}$.