When determining the limit of a sequence

$$\lim_{n\to\infty}\frac{x_n}{y_n} \quad \text{where} \quad \lim_{n\to\infty} x_n = \infty \quad \lim_{n\to\infty} y_n = \infty$$

the following theorem due to O. Stolz may be used to advantage

Let $\lim_{n\to\infty} y_n = \infty$ and let some index N exist such that

$y_{n+1} > y_n$ whenever $n > N$. Then

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}}$$

provided that the right-hand-side limit exists (finite or infinite)

<u>Proof</u>

Assume at first that the limit in question is finite:

$$\lim_{n\to\infty} \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = l$$

Then, for any $\varepsilon > 0$, there exists an index N such that

for every $n > N$ we have $\left| \frac{y_n - y_{n-1}}{x_n - x_{n-1}} - l \right| < \frac{\varepsilon}{2}$, that is,

$$\frac{\varepsilon}{2} - l < \frac{y_n - y_{n-1}}{x_n - x_{n-1}} < \frac{\varepsilon}{2} + l$$

In other words, for any $n > N$, we have

$$l - \frac{\varepsilon}{2} < \frac{x_{N+1} - x_N}{y_{N+1} - y_N} < l + \frac{\varepsilon}{2}$$

$$l - \frac{\varepsilon}{2} < \frac{x_{N+2} - x_{N+1}}{y_{N+2} - y_{N+1}} < l + \frac{\varepsilon}{2}$$

$$\vdots$$

$$l - \frac{\varepsilon}{2} < \frac{x_{n-1} - x_{n-2}}{y_{n-1} - y_{n-2}} < l + \frac{\varepsilon}{2}$$

$$l - \frac{\varepsilon}{2} < \frac{x_n - x_{n-1}}{y_n - y_{n-1}} < l + \frac{\varepsilon}{2}$$

Since, starting from an index, we have $y_{n+1} > y_n$, we may assume that all the denominators of the fractions are positive and so

$$\left(l-\frac{\varepsilon}{2}\right)(y_{N+1}-y_N) < x_{N+1}-x_N < \left(l+\frac{\varepsilon}{2}\right)(y_{N+1}-y_N)$$

$$\left(l-\frac{\varepsilon}{2}\right)(y_{N+2}-y_{N+1}) < x_{N+2}-x_{N+1} < \left(l+\frac{\varepsilon}{2}\right)(y_{N+2}-y_{N+1})$$

$$\vdots$$

$$\left(l-\frac{\varepsilon}{2}\right)(y_{n-1}-y_{n-2}) < x_{n-1}-x_{n-2} < \left(l+\frac{\varepsilon}{2}\right)(y_{n-1}-y_{n-2})$$

$$\left(l-\frac{\varepsilon}{2}\right)(y_{n}-y_{n-1}) < x_{n}-x_{n-1} < \left(l+\frac{\varepsilon}{2}\right)(y_{n}-y_{n-1})$$

Adding up all these inequalities yields

$$\left(l-\frac{\varepsilon}{2}\right)(y_n-y_N)<x_n-x_N<\left(l+\frac{\varepsilon}{2}\right)(y_n-y_N)$$

and then

$$\left(l-\frac{\varepsilon}{2}\right)<\frac{x_n-x_N}{y_n-y_N}<\left(l+\frac{\varepsilon}{2}\right)$$

or

$$\left|\frac{x_n-x_N}{y_n-y_N}-l\right|<\frac{\varepsilon}{2}$$

We will now show that there is an index $N^{'}$ such that,

for $n > N^{'}$ we have $\left|\frac{x_n}{y_n} - l\right| < \varepsilon$. To this end, let us calculate

$$\left|\frac{x_n}{y_n} - l\right| = \left|\frac{x_n - ly_n}{y_n}\right| = \left|\frac{x_n - ly_n + x_N - x_N + ly_N - ly_N}{y_n}\right| =$$

$$= \left|\frac{x_N - ly_N}{y_n} + \frac{x_n - x_N - ly_n + ly_N}{y_n}\right| \leq \left|\frac{x_N - ly_N}{y_n}\right| + \left|\frac{x_n - x_N - ly_n + ly_N}{y_n}\right| =$$

$$= \left|\frac{x_N - ly_N}{y_n}\right| + \left|\frac{(y_n - y_N)\left(x_n - x_N - l(y_n + y_N)\right)}{(y_n - y_N)\, y_n}\right| =$$

$$=\left|\frac{x_N - l y_N}{y_n}\right| + \left|\frac{(y_n - y_N)}{y_n}\right| \left|\frac{(x_n - x_N - l(y_n + y_N))}{(y_n - y_N)}\right| =$$

$$=\left|\frac{x_N - l y_N}{y_n}\right| + \left|\frac{y_n - y_N}{y_n}\right| \left|\frac{x_n - x_N}{y_n - y_N} - l\right| = \left|\frac{x_N - l y_N}{y_n}\right| + \left|1 - \frac{y_N}{y_n}\right| \left|\frac{x_n - x_N}{y_n - y_N} - l\right| \leq$$

$$\leq \left|\frac{x_N - l y_N}{y_n}\right| + \left|\frac{x_n - x_N}{y_n - y_N} - l\right|$$

The last inequality on the previous slide follows from the fact that, since $\lim_{n\to\infty} y_n = \infty$, we have an index N'' such that, for $n > N''$, y_n is positive and since $\{y_i\}_{i=1}^{\infty}$ is also increasing from a certain index N''', choosing $N'' > N'''$ we can achieve that, for $n > N''$, we have $\left|1 - \dfrac{y_N}{y_n}\right| \leq 1$

Thus we have $\left|\dfrac{x_n}{y_n} - l\right| \leq \left|\dfrac{x_N - l y_N}{y_n}\right| + \left|\dfrac{x_n - x_N}{y_n - y_N} - l\right|$ so that

$\left|\dfrac{x_n}{y_n} - l\right| \leq \left|\dfrac{x_N - l y_N}{y_n}\right| + \dfrac{\varepsilon}{2}$. However, since $\lim_{n\to\infty} y_n = \infty$, there is an index N'''' such that $\left|\dfrac{x_N - l y_N}{y_n}\right| < \dfrac{\varepsilon}{2}$ for $n > N''''$.

It is now clear that, if we put $N' = \max\{N'', N''', N''''\}$, we have $\left|\frac{x_n}{y_n} - l\right| < \varepsilon$ for $n > N'$. This proves the theorem for the event that

$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$ is finite. If it is infinite, say, $\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \infty$, then

for sufficiently large indices *n*, we have $x_n - x_{n-1} > y_n - y_{n-1}$,

which means that $\lim_{n\to\infty} x_n = \infty$ and $\{x_i\}_{i=1}^{\infty}$ is increasing up from a

certain index. Then we can apply the already proved finite version

of the theorem to the reversed fraction obtaining

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \lim_{n\to\infty} \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = 0$$

which means that $\lim_{n\to\infty} \frac{x_n}{y_n} = \infty$

Example

Calculate $\lim_{n\to\infty} \frac{a^n}{n}$ where $a > 1$.

Using Stoltz's theorem, we have

$$\lim_{n\to\infty} \frac{a^n}{n} = \lim_{n\to\infty} \frac{a^n - a^{n-1}}{n-(n-1)} = \lim_{n\to\infty} a^n - a^{n-1} = \lim_{n\to\infty} a^n\left(1-\frac{1}{a}\right) = \infty$$

Example

Calculate $\lim_{n\to\infty}\frac{1^k+2^k+\cdots+n^k}{n^{k+1}}$

Using Stoltz's theorem, we have

$$\lim_{n\to\infty}\frac{1^k+2^k+\cdots+n^k}{n^{k+1}}=\lim_{n\to\infty}\frac{n^k}{n^{k+1}-(n-1)^{k+1}}$$

But $(n-1)^{k+1}=n^{k+1}-\binom{k+1}{1}n^k+\cdots$ and so

$n^{k+1}-(n-1)^{k+1}=\binom{k+1}{1}n^k+\cdots$ Substituting then yields

$$\lim_{n\to\infty}\frac{n^k}{n^{k+1}-(n-1)^{k+1}}=\lim_{n\to\infty}\frac{n^k}{(k+1)n^k+\cdots}=\frac{1}{k+1}$$

The previous example can be used to calculate the definite integral

$$\int_0^x t^k dt$$

Partitioning the interval [0, *x*] into *n* equal subintervals we can set up the upper integral sum $U\left(t^k, 0, x, n\right)$

$$U\left(t^k,0,x,n\right)=\frac{x}{n}\left(\left(\frac{x}{n}\right)^k+\left(\frac{2x}{n}\right)^k+\cdots+\left(\frac{(n-1)x}{n}\right)^k+\left(\frac{nx}{n}\right)^k\right)=$$

$$U\left(t^k,0,x,n\right)=\frac{x}{n}\left(\left(\frac{x}{n}\right)^k+\left(\frac{2x}{n}\right)^k+\cdots+\left(\frac{(n-1)x}{n}\right)^k+\left(\frac{nx}{n}\right)^k\right)=$$

$$U\left(t^k,0,x,n\right)=x^{k+1}\left(\frac{1^k+2^k+\cdots+n^k}{n^{k+1}}\right)$$

Using Stoltz's theorem we can now write

$$\int_0^x t^k dt = \lim_{n\to\infty} U\left(t^k, 0, x, n\right) = x^{k+1} \lim_{n\to\infty}\left(\frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}\right) = \frac{x^{k+1}}{k+1}$$