To any sequence $\{a_n\}_{n=1}^{\infty}$ we can assign a sequence $\{s_n\}_{n=1}^{\infty}$ with terms defined as

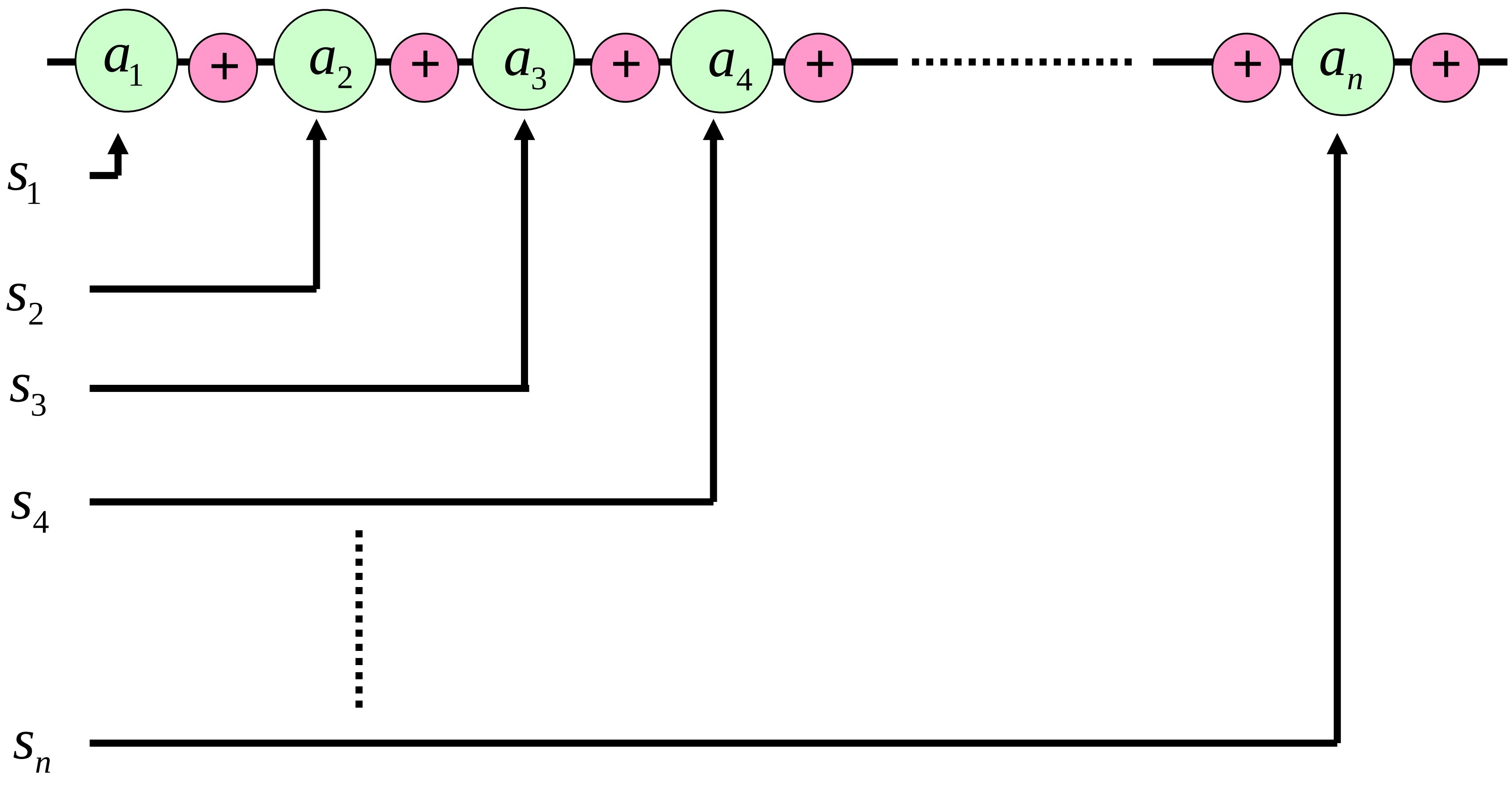
$$s_n = a_1 + a_2 + \cdots + a_n$$

and an expression

$a_1 + a_2 + \cdots + a_n + \cdots$ also denoted $\sum_{n=1}^{\infty} a_n, \sum_{1}^{\infty} a_n, \sum a_n$

which is called an (infinite) (number) **series** with terms a_n

The sequence $\{s_n\}_{n=1}^{\infty}$ is called the **sequence of partial sums** of $\sum_{n=1}^{\infty} a_n$



a_1
+
a_2
+
a_3
+
a_4
+
+
a_n
+
s_1
s_2
s_3
s_4
s_n

We say that a series $\sum a_n$ **converges** and has a finite sum s, which is denoted $a_1 + a_2 + \cdots + a_n + \cdots = s$ or $\sum a_n = s$ if $s = \lim_{n\to\infty} s_n$

If $\sum a_n$ does not converge, we say that it **diverges** or is divergent and either $\lim_{n\to\infty} s_n = \infty$ or the limit does not exist, in which case we say that $\sum a_n$ **oscillates**.

Example

The series $\sum_{1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}$

converges and adds up to 1

$$s_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$s_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$$

Example

The series $\sum_{n=1}^{\infty}(-1)^{n+1} = 1+(-1)+(-1)+\cdots = 1-1+1-\cdots$

oscillates since we have $s_{2k+1} = 1$ and $s_{2k} = 0$

Necessary condition of convergence

If a series $\sum a_n$ converges, then $\lim\limits_{n \to \infty} a_n = 0$

Example

The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$ is called **harmonic**.

It diverges even if, clearly, $\displaystyle\lim_{n\to\infty}\frac{1}{n}=0$

$$s_n=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

$$s_{2n}-s_n=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\geq n\frac{1}{2n}=\frac{1}{2} \qquad (*)$$

Suppose $\{s_n\}_{n=1}^{\infty}$ converges, then so does $\{s_{2n}\}_{n=1}^{\infty}$ and thus

$\displaystyle\lim_{n\to\infty}s_{2n}-s_n=0$, which is not possible due to (*)

Geometric series

A geometric series $\sum_{n=0}^{\infty} aq^n = a + aq + aq^2 + \cdots + aq^n + \cdots$

where $a \neq 0$ converges if $\left| q < 1 \right|$ otherwise it diverges.

$$s_n = a + aq + aq^2 + \cdots + aq^n$$

$$qs_n = qa + aq^2 + aq^3 + \cdots + aq^{n+1}$$

$$s_n - qs_n = a - aq^{n+1}$$

$$s_n = a\frac{1-q^{n+1}}{1-q} =$$

$$\lim_{n\to\infty} s_n = \frac{a}{1-q} \text{ for } |\mathrm{q}| < 1$$

$$\lim_{n\to\infty} s_n = \infty \text{ or } -\infty \text{ for } q > 1$$

$$\lim_{n\to\infty} s_n \text{ does not exist for } q < -1$$

For a positive integer p, both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=p}^{\infty} a_n$ either converge or diverge at the same time. If they converge, then

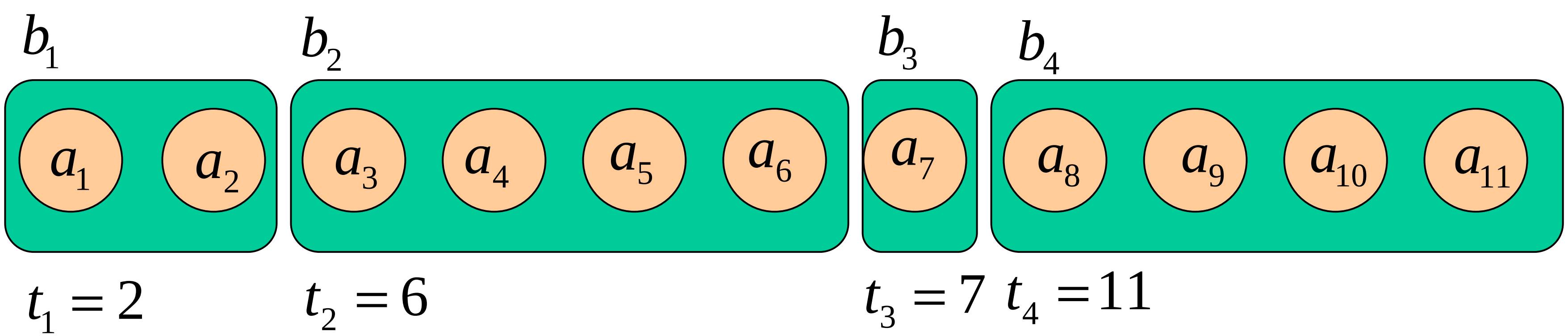
$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{p-1} + \sum_{n=p}^{\infty} a_n$$

For a non-zero real k, both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} ka_n$ either converge or diverge at the same time. If they converge, then

$$\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n$$

A convergent series has the property that its neighbouring terms can be associated without changing the sum of the series, which is what the following theorem says:

Let $\sum_{n=1}^{\infty} a_n = s$. Let further $\{t_n\}_{n=1}^{\infty}$ be an increasing sequence of natural numbers and $t_0 = -1$. Let $b_n = a_{t_{n-1}+1} + a_{t_{n-1}+2} + \cdots + a_{t_n}$ then $\sum_{n=1}^{\infty} b_n = s$



Note that a divergent series may become convergent after its terms are associated as the following example shows

$$1-1+1-1+1-\cdots \text{ diverges}$$

$$(1-1)+(1-1)+(1-1)+\cdots \text{ converges}$$

Sum and difference of series

For two series $\sum a_n$ and $\sum b_n$ we define their sum or difference as the sum or difference of their corresponding terms, that is,

$$\sum a_n + \sum b_n = \sum (a_n + b_n)$$
$$\sum a_n - \sum b_n = \sum (a_n - b_n)$$

Let $\sum a_n$ and $\sum b_n$ be convergent series and let

$\sum a_n = a$ and $\sum b_n = b$, then

$$\sum (a_n + b_n) = a + b$$
$$\sum (a_n - b_n) = a - b$$

Series with positive (non-negative) terms

For a series with positive (non-negative) terms the following assertions are obvious:

- The sequence of partial sums of a series with positive (non-negative) terms is increasing (non-decreasing)
- If the sequence of partial sums of a series with positive (non-negative) terms has an upper bound, the series converges
- A series with positive (non-negative) terms cannot oscillate

Example

The series $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges for $a > 1$.

The sequence of partial sums $s_n = \frac{1}{1^a} + \frac{1}{2^a} + \cdots + \frac{1}{n^a}$ is increasing.

We will show that it is bounded above. Let n be an arbitrary index and k such a natural number that $n < 2^k$. Then

$$s_n = \frac{1}{1^a} + \frac{1}{2^a} + \cdots + \frac{1}{n^a} \leq \frac{1}{1^a} + \frac{1}{2^a} + \cdots + \frac{1}{\left(2^k - 1\right)^a} =$$

$$=\frac{1}{1^a}+\left(\frac{1}{2^a}+\frac{1}{3^a}\right)+\left(\frac{1}{4^a}+\frac{1}{5^a}+\frac{1}{6^a}+\frac{1}{7^a}\right)+\cdots$$

$$\cdots+\left(\frac{1}{\left(2^{k-1}\right)^a}+\frac{1}{\left(2^{k-1}+1\right)^a}+\cdots+\frac{1}{\left(2^k-1\right)^a}\right)=$$

$$=1+\frac{2}{2^a}+\frac{4}{4^a}+\cdots+\frac{2^{k-1}}{\left(2^{k-1}\right)^a}=1+\frac{1}{2^{a-1}}+\frac{1}{4^{a-1}}+\cdots+\frac{1}{\left(2^{k-1}\right)^{a-1}}=$$

$$=1+\frac{1}{2^{a-1}}+\left(\frac{1}{2^{a-1}}\right)^2+\cdots+\left(\frac{1}{2^{a-1}}\right)^{k-1}=\frac{1-\left(\frac{1}{2^{a-1}}\right)^k}{1-\frac{1}{2^{a-1}}}<\frac{1}{1-\frac{1}{2^{a-1}}}=\frac{2^{a-1}}{2^{a-1}-1}$$

The previous formulas hold if $0 < \dfrac{1}{2^{a-1}} < 1$, which means that

$2^{a-1} > 1$ However, the last inequality holds if $a - 1 > 0$ or $a > 1$

which is the assumption of the assertion.