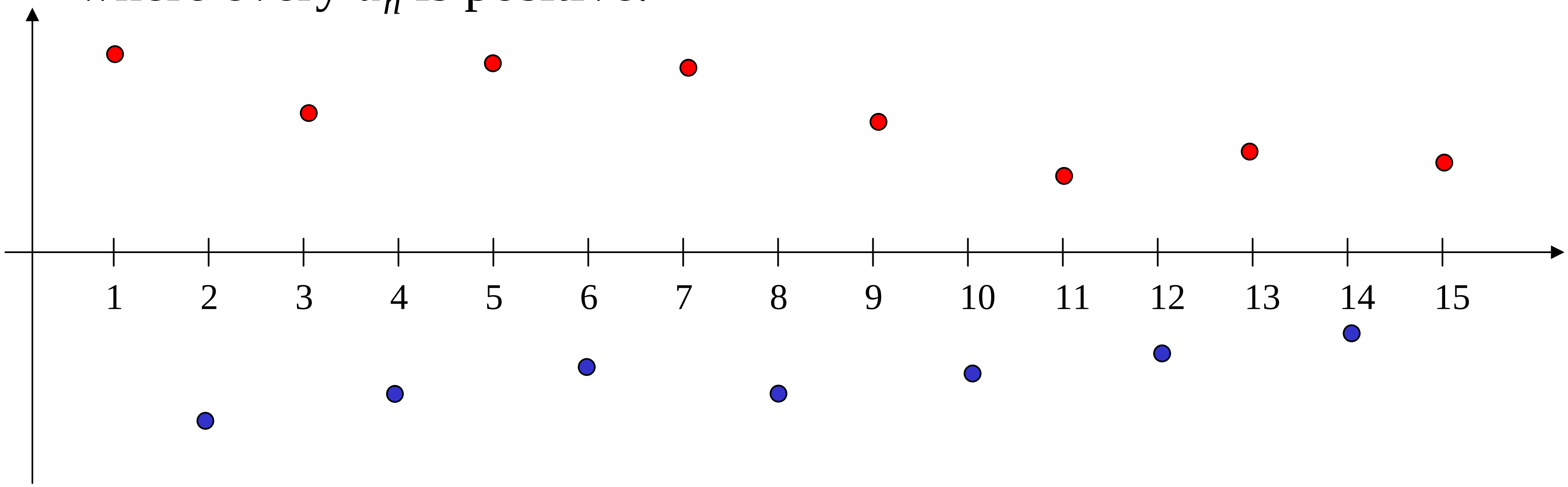
Alternating series

We refer to a series as an alternating series if its terms' signs alternate, that is, the sequence looks like this

$$\sum_{n=1}^{\infty}(-1)^n a_n = a_1 - a_2 + a_3 - \cdots$$

where every a_n is positive.



The convergence of an alternating series is governed by a theorem proved by Leibnitz

If $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence of positive numbers

which converges to zero, then the alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n = a_1 - a_2 + a_3 - \cdots$$

converges and its sum s satisfies the inequalities

$$a_1 - a_2 < s < a_1$$

Examples

The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges

In fact, we even can calculate the sum of this series, which is $\ln 2$

So does the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^a}$ where $a > 0$

Absolutely convergent series

Let $\sum_{n=1}^{\infty} a_n$ be a series. We say that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent or that it absolutely converges if the series $\sum_{n=1}^{\infty} |a_n|$ converges

An absolutely convergent series is convergent

Relatively convergent series

Let $\sum_{n=1}^{\infty} a_n$ be a series. We say that $\sum_{n=1}^{\infty} a_n$ is relatively convergent or that it relatively converges if it converges, but the series $\sum_{n=1}^{\infty} |a_n|$ diverges

Examples

The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges relatively since it

converges due to the Leibnitz theorem, but the series

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

which is known as the harmonic series, diverges (as can be

established, for example, by the integral criterion).

Let $\sum_{n=1}^{\infty} a_n$ be a relatively convergent series. Put

$$p_n = \begin{cases} 0 \text{ for } a_n < 0 \\ a_n \text{ for } a_n \geq 0 \end{cases} \qquad q_n = \begin{cases} 0 \text{ for } a_n > 0 \\ -a_n \text{ for } a_n \leq 0 \end{cases}$$

Then $\lim\limits_{n \to \infty} \sum p_n = \infty$ and $\lim\limits_{n \to \infty} \sum q_n = \infty$

$$\lim_{n \to \infty} p_n = 0 \quad \text{and} \quad \lim_{n \to \infty} q_n = 0$$

Proof

$\sum_{n=1}^{\infty} a_n$ converges and so $\lim_{n\to\infty} a_n = 0$ and thus also $\lim_{n\to\infty} |a_n| = 0$

Since $0 \le p_n \le |a_n|$ and $0 \le q_n \le |a_n|$, this proves that

$\lim_{n\to\infty} q_n = 0$ and $\lim_{n\to\infty} q_n = 0$

If $\sum p_n$ and $\sum q_n$ both were convergent, then also their sum

$\sum (p_n + q_n) = \sum |a_n|$ would be convergent.

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} p_n = p \text{ and } \lim_{n\to\infty} \sum_{n=1}^{\infty} q_n = \infty \Rightarrow \lim_{n\to\infty} \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \left(\sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n \right) = -\infty$$

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} p_n = \infty \text{ and } \lim_{n\to\infty} \sum_{n=1}^{\infty} q_n = a \Rightarrow \lim_{n\to\infty} \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \left(\sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n \right) = \infty$$

Rearranging the terms of a series

Let us have two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{i_n}$

We say that $\sum_{n=1}^{\infty} a_{i_n}$ has been formed from $\sum_{n=1}^{\infty} a_n$ by a term rearrangement or is a rearranged $\sum_{n=1}^{\infty} a_n$ if in the sequence

$$n_1, n_2, n_3, \dots$$

each natural number occurs exactly once

Example

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + \cdots$$

$$a_2 + a_1 + a_4 + a_3 + a_6 + a_5 + a_8 + a_7 + a_{10} + a_9 + a_{12} + a_{11} + a_{14} + \cdots$$

Let $\sum_{n=1}^{\infty} a_n$ absolutely converge. Then no rearrangement of its terms can change its sum.

The following remarkable theorem is due to Riemann:

Let $\sum_{n=1}^{\infty} a_n$ be a relatively convergent series and let S be an arbitrary real number or $S = \infty$. Then the terms of $\sum_{n=1}^{\infty} a_n$ can be rearranged so that

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} a_n = S$$

Example instead of proof

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\frac{1}{13}-\frac{1}{14}+\frac{1}{15}\cdots$$

$$\boxed{S=1.5}$$

$$1+\frac{1}{3}+\frac{1}{5}=1.53333>1.5$$

$$1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}=1.03333<1.5$$

$$1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}=1.52180>1.5$$

$$1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}-\frac{1}{4}=1.2718>1.5$$

$$1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}-\frac{1}{4}+\frac{1}{17}+\frac{1}{19}+\frac{1}{21}+\frac{1}{23}+\frac{1}{25}=1.51435>1.5$$

$$1.5$$