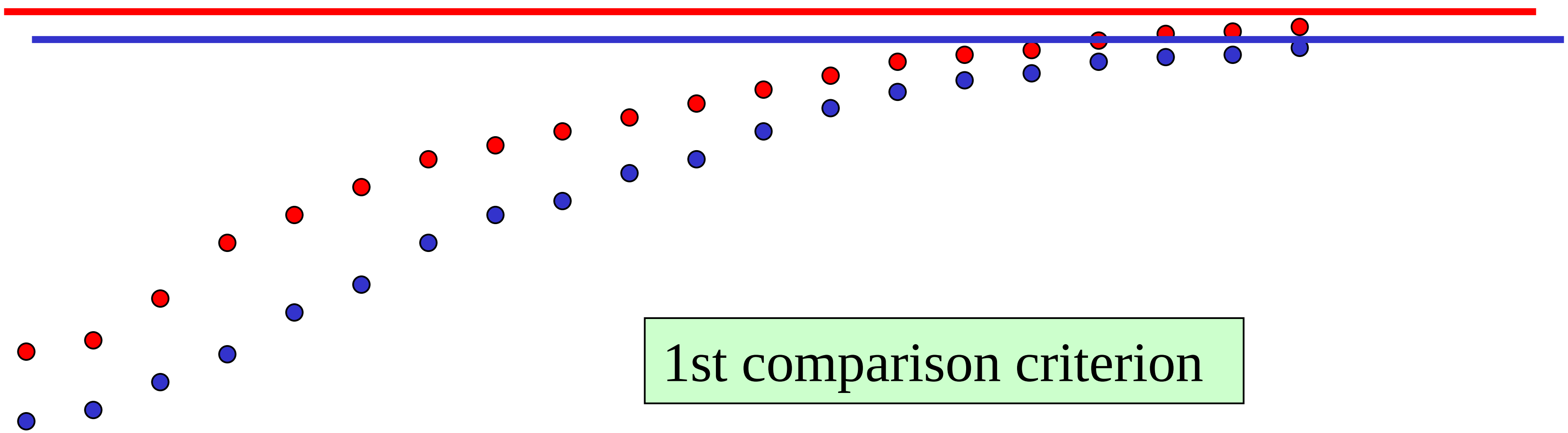
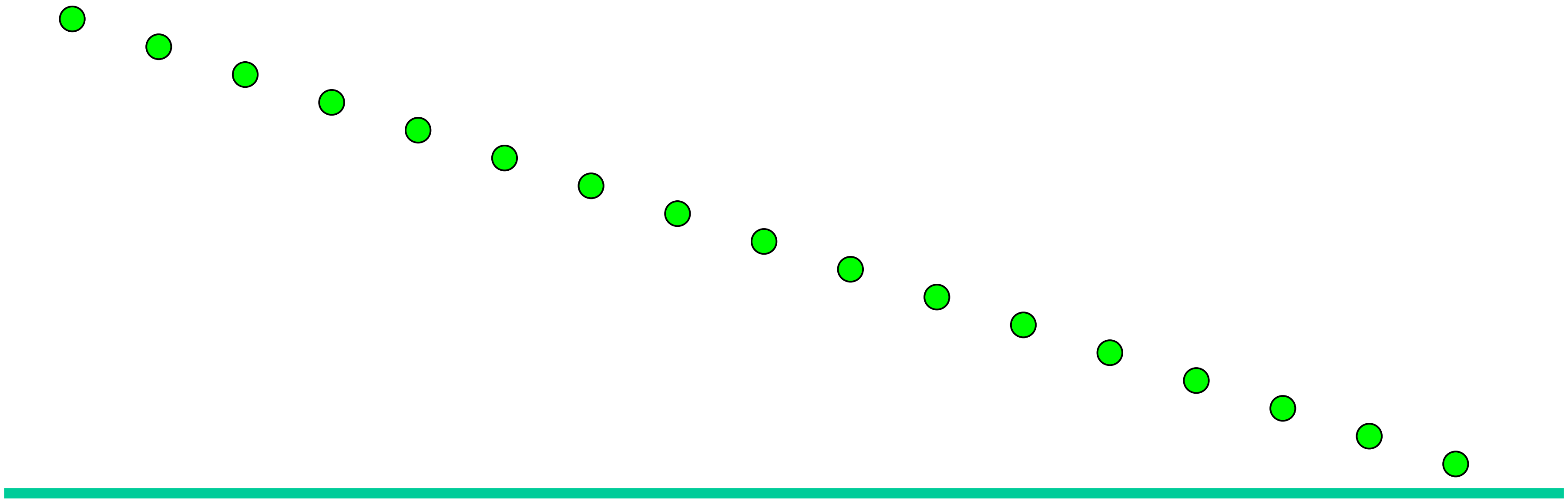
Let $\sum a_n, \sum b_n$ be series with non-negative terms. Let, for almost every n, $a_n \leq b_n$. Then if $\sum b_n$ converges, so does $\sum a_n$. If $\sum a_n$ diverges, so does $\sum b_n$.



1st comparison criterion

The proof of the previous assertion uses the following fact, which could also be proved about sequences:

If a sequence is non-decreasing and bounded above or non-increasing and bounded below, it converges.



Example

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$$\frac{1}{n^2} < \frac{1}{n(n-1)} \text{ for } n \geq 2$$

$$\frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3} + \cdots + \frac{1}{n\cdot(n-1)} =$$

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{n-1} + \frac{1}{n-1} - \frac{1}{n} = 1 - \frac{1}{n} \to 1$$

Let $\sum a_n, \sum b_n$ be series with non-negative terms. Let, for almost every n, $\dfrac{a_n}{a_{n+1}} \le \dfrac{b_n}{b_{n+1}}$. Then if $\sum b_n$ converges, so does $\sum a_n$.

If $\sum a_n$ diverges, so does $\sum b_n$.

The previous comparison tests give rise to various tests based on comparison with a geometris series

$$a + aq + aq^2 + \cdots + aq^n + \cdots$$

which is known to be convergent if $q < 1$.

Root test

Let $\sum a_n$ be a series with non-negative terms. Let, for almost every *n*, $0 \le \sqrt[n]{a_n} \le q, \quad q < 1$, then the series converges. If, on the contrary, we have $1 \le \sqrt[n]{a_n}$ for almost every *n*, then $\sum a_n$ diverges.

This can also be expressed in a different way:

Let $\sum a_n$ be a series with non-negative terms.

Let, $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = q.$ If $q < 1$, then the series converges.

If $q > 1$, then it diverges.

Example

Find out whether the series $\sum \left(\dfrac{3n^3 - n}{4n^3 + n^2 + 7} \right)^n$ is convergent.

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n^3 - n}{4n^3 + n^2 + 7} \right)^n} = \lim_{n\to\infty} \left(\frac{3n^3 - n}{4n^3 + n^2 + 7} \right) = \frac{3}{4} < 1$$

Quotient test

Let $\sum a_n$ be a series with positive terms. Let $\frac{a_{n+1}}{a_n} \leq q$ for almost every n and $q < 1$. Then $\sum a_n$ converges.

If $\frac{a_{n+1}}{a_n} \geq 1$ for almost every n, then $\sum a_n$ diverges.

Quotient test - its limit variant

Let $\sum a_n$ be a series with positive terms. Let $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = q$ and $q < 1$. Then $\sum a_n$ converges. If $q > 1$, it diverges

Example

Is the series $\sum \frac{e^n}{n!}$ convergent?

$$\lim_{n\to\infty} \frac{\dfrac{e^{n+1}}{(n+1)!}}{\dfrac{e^n}{n!}} = \lim_{n\to\infty} \frac{e^{n+1} n!}{(n+1)! e^n} = \lim_{n\to\infty} \frac{e}{(n+1)} = 0$$

Integral test

Let a function $f(x)$ be positive and non-increasing on $[1, \infty)$.

If $f(n) = a_n$ for $n = 1, 2, 3, \dots$, then the series $\sum a_n$ converges

exactly when the integral $I = \int_1^\infty f(x)\,dx$ is convergent.

<u>Example</u>

The series $\sum_{n=2}^{\infty} \frac{1}{n \ln^a n}$ converges for $a > 1$ and diverges for $a \le 1$.

The function $f(x) = \frac{1}{x \ln^a x}$ is positive and decreasing on $[1, \infty)$

∞ if $a < 1$

0 if $a > 1$

if a < 0.

$$\int_2^{\infty} \frac{dx}{x \ln^a x} = \lim_{b\to\infty} \left[\frac{\ln^{1-a} x}{1-a} \right]_{\sqrt{2}}^{b} = \lim_{b\to\infty} \left(\frac{\ln^{1-a} b}{1-a} - \frac{\ln^{1-a} \sqrt{2}}{1-a} \right)$$

finite number

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b\to\infty} \left[\ln(\ln x) \right]_2^{b} = \lim_{b\to\infty} \left(\ln(\ln b) - \ln(\ln 2) \right) = \infty$$