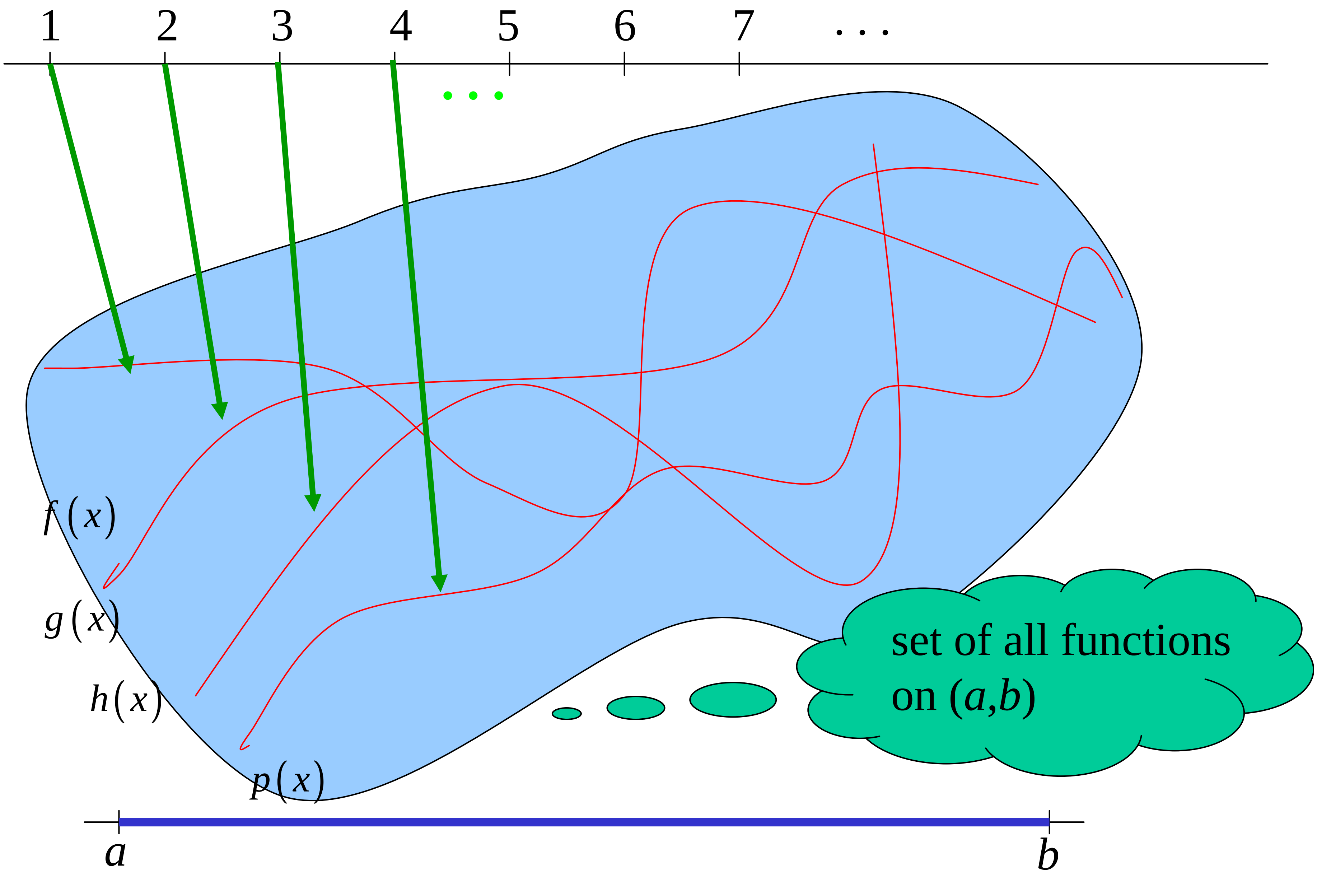
Function sequences



Function sequence

$$\{f_n(x)\}_{n=1}^{\infty} = f_1(x), f_2(x), f_3(x), \ldots$$

where $x \in D \subseteq R$

Function series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \ldots \qquad \text{where } x \in D \subseteq R$$

$$s_n(x) = \sum_{n=1}^{n} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \ldots + f_n(x)$$

Point-wise convergence

If for a function sequence $\{f_n(x)\}_{n=1}^{\infty}$ or a function series $\sum_{n=1}^{\infty} f_n(x)$ at each $a \in D$ the number sequence $\{f_n(a)\}_{n=1}^{\infty}$ or number series $\sum_{n=1}^{\infty} f_n(a)$ converges to $f(a)$ or $s(a)$, then we say that they converge point-wise in D.

Thus point-wise convergence defines new functions $f(x)$ and $s(x)$

We write

$$\boxed{\lim_{n\to\infty} f_n(x) = f(x)}$$

$$\boxed{\sum_{n=1}^{\infty} f_n(x) = s(x)}$$

Example

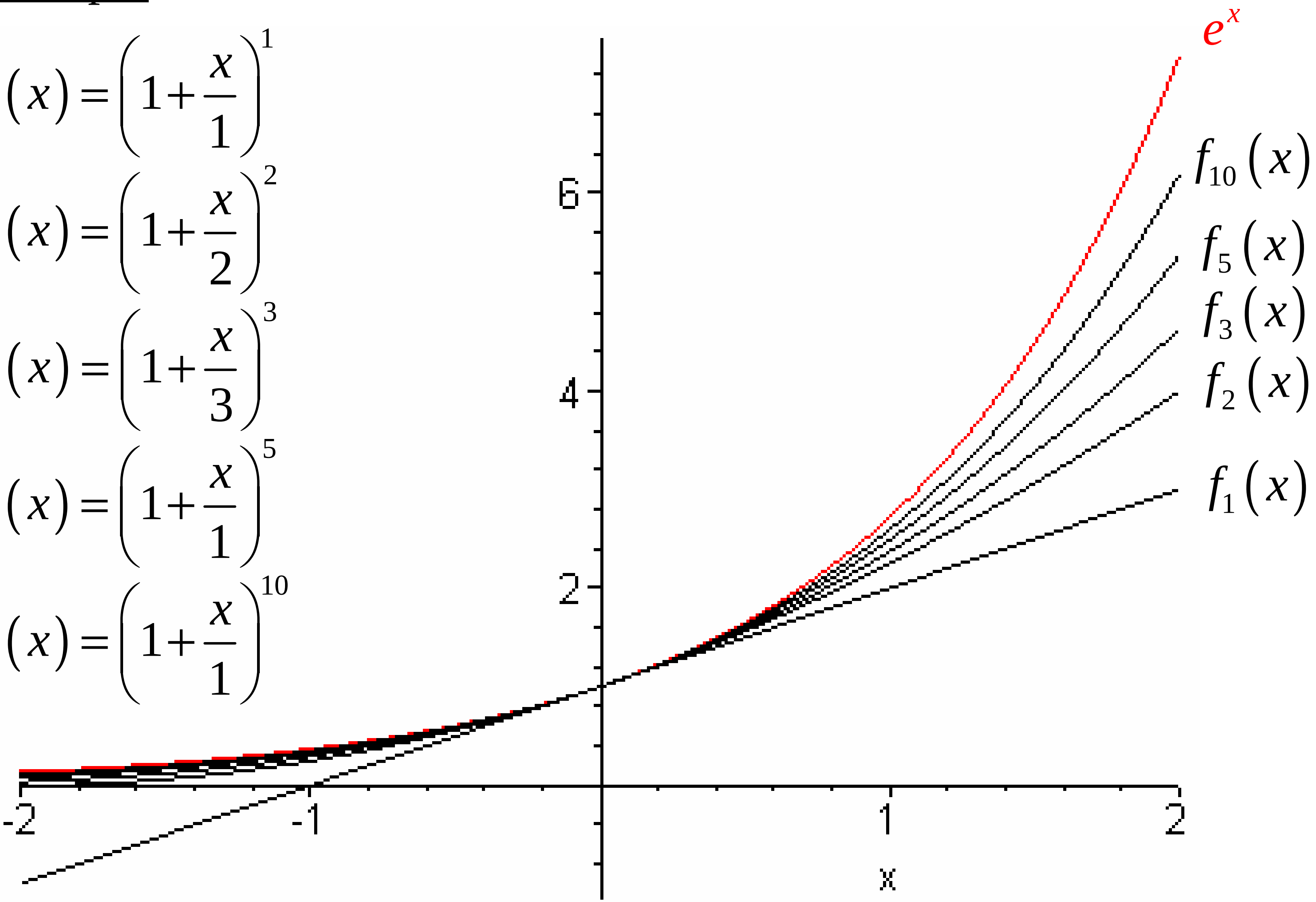
$$f_1(x)=\left(1+\frac{x}{1}\right)^1$$

$$f_2(x)=\left(1+\frac{x}{2}\right)^2$$

$$f_3(x)=\left(1+\frac{x}{3}\right)^3$$

$$f_5(x)=\left(1+\frac{x}{1}\right)^5$$

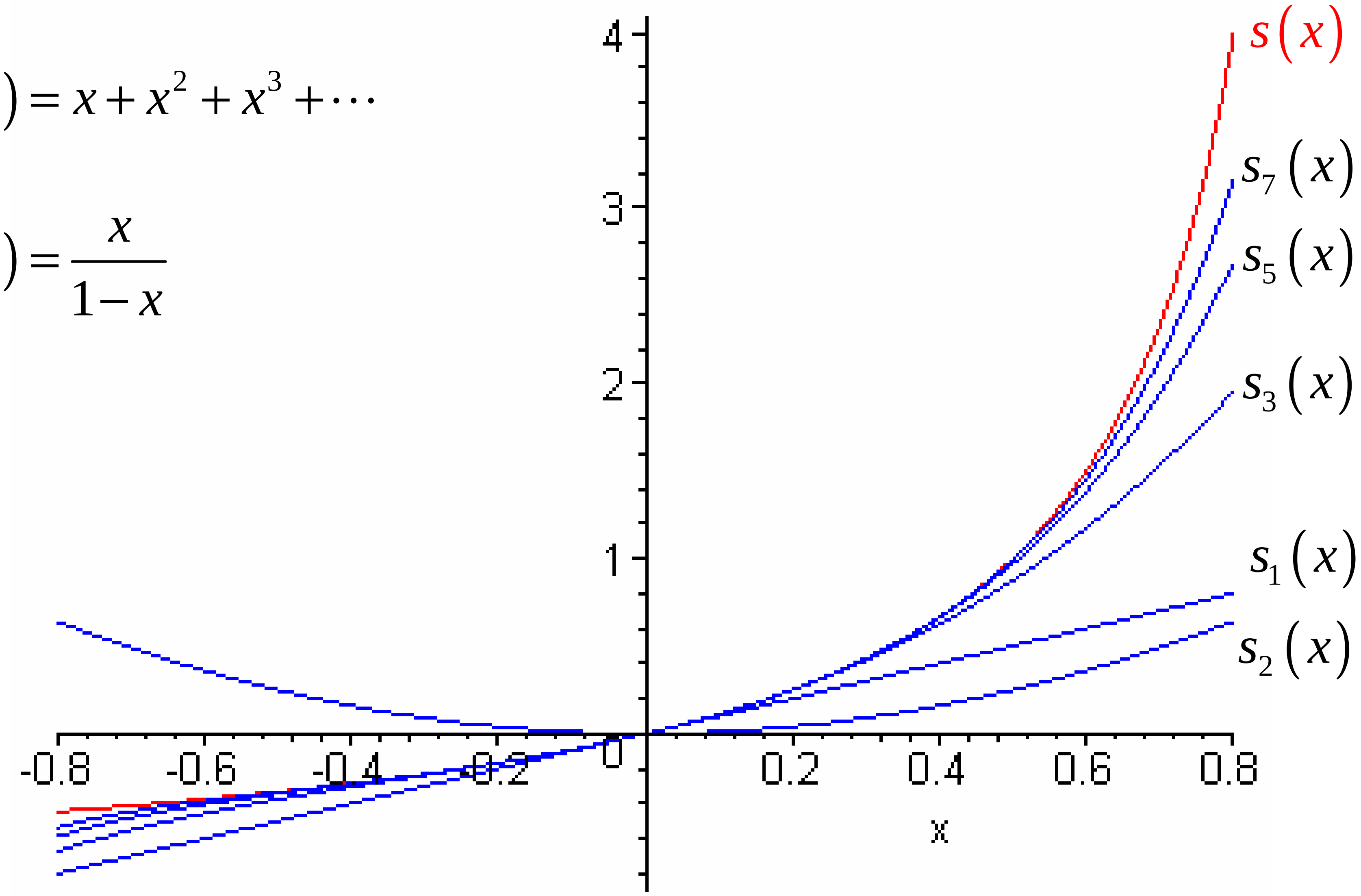
$$f_{10}(x)=\left(1+\frac{x}{1}\right)^{10}$$



Example

$$s(x) = x + x^2 + x^3 + \cdots$$

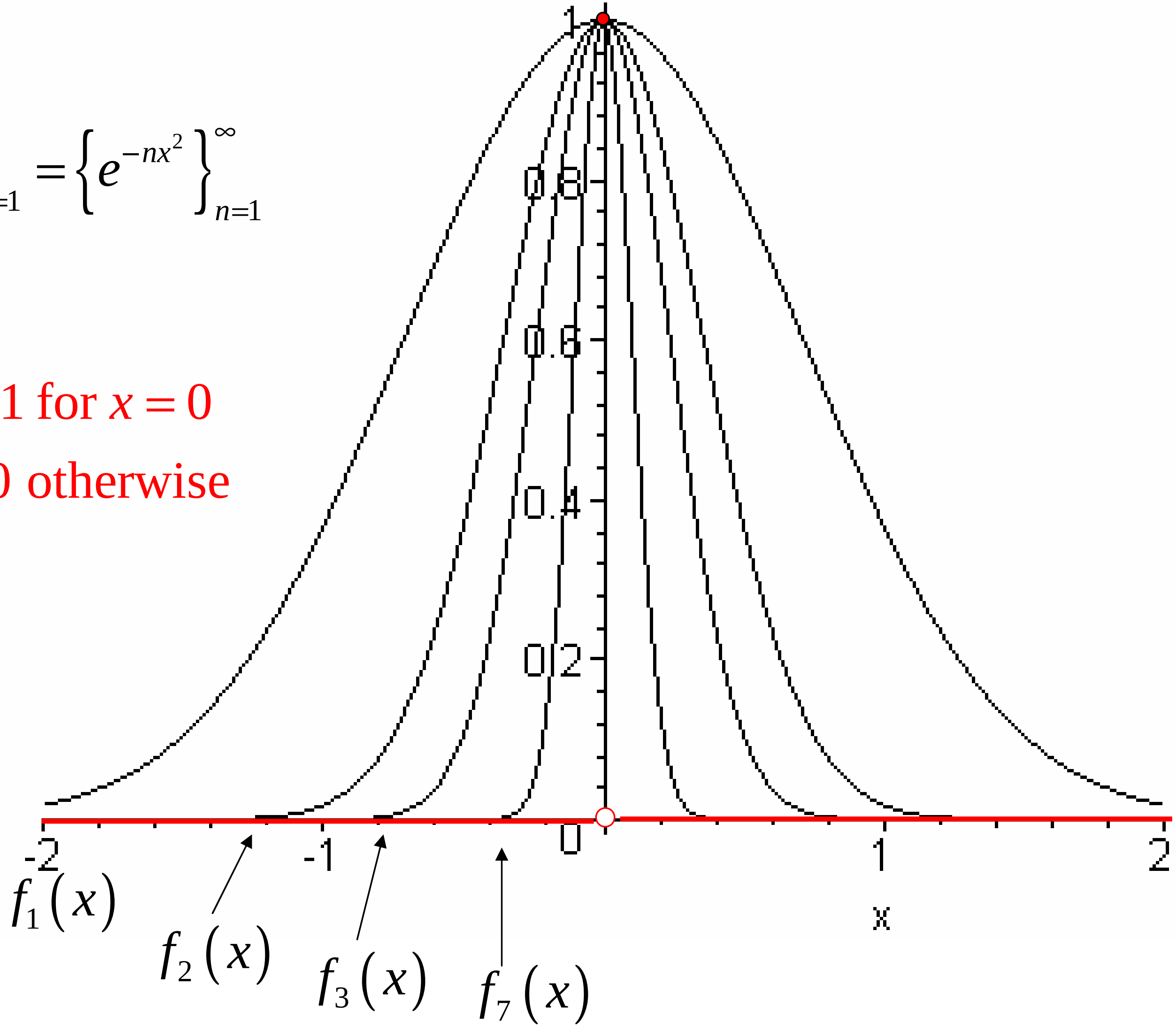
$$s(x) = \frac{x}{1-x}$$



Example

$$\left\{ f_n(x) \right\}_{n=1}^{\infty} = \left\{ e^{-nx^2} \right\}_{n=1}^{\infty}$$

$$f(x) = \begin{cases} 1 \text{ for } x = 0 \\ 0 \text{ otherwise} \end{cases}$$

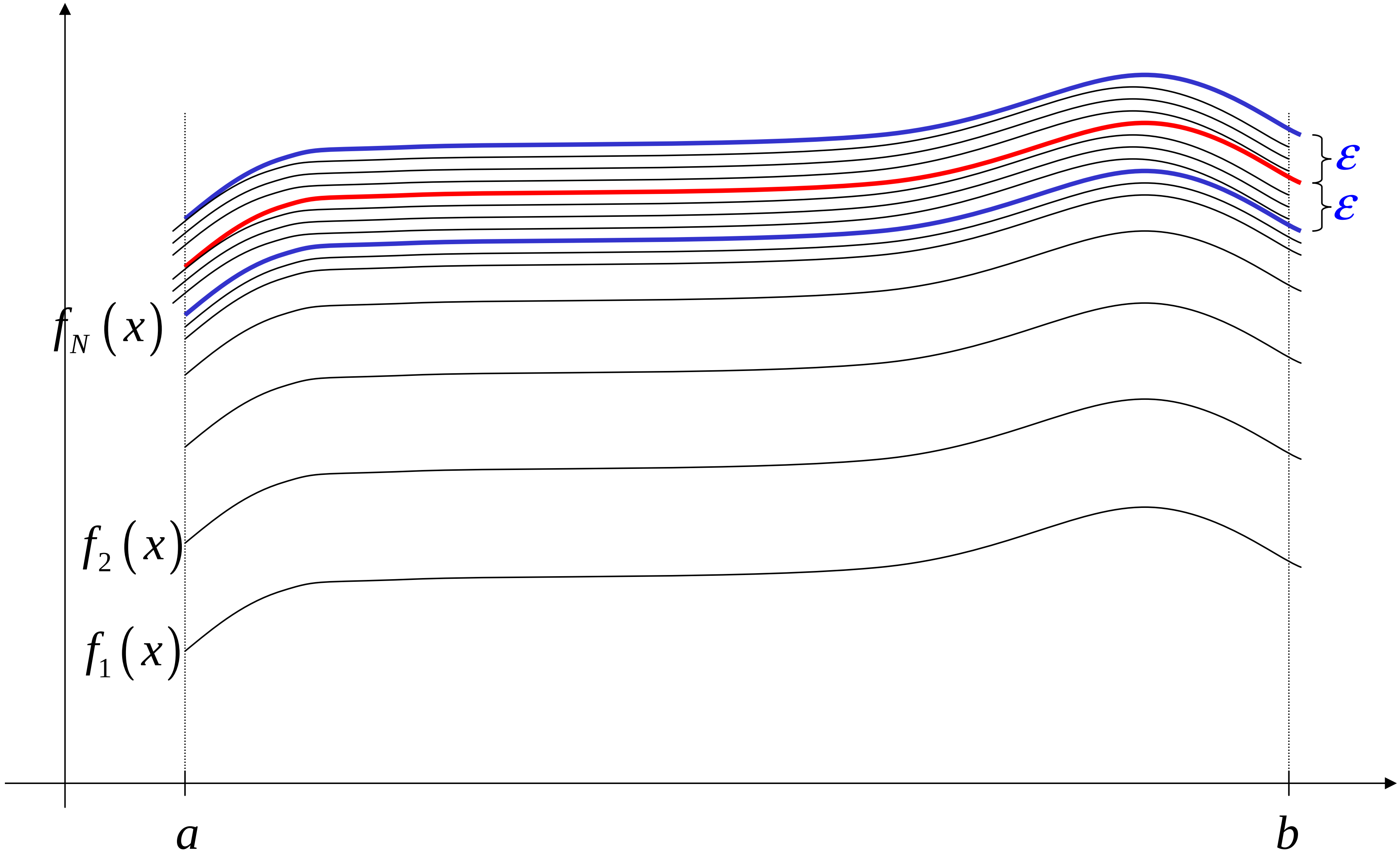


- Note that, in the last example, although the sequence functions are all continuous, the limit function is not.

- Thus point-wise convergence may define non-continuous using continuous ones.

- We shall examine such sequences of continuous functions that only produce continuous results.

Uniform convergence



A function sequence $\left\{ f_n\left(x\right)\right\}_{n=1}^{\infty}$ converges uniformly to a function $f\left(x\right)$ on $[a,b]$ if, for every ε, there exists an index N such that, for every $x \in \left[a,b\right]$ and for every $n > N$, we have

$$f\left(x\right)-\varepsilon < f_n\left(x\right) < f\left(x\right)+\varepsilon \text{ or } \left|f\left(x\right)-f_n\left(x\right)\right| < \varepsilon$$

A function series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a function $s(x)$ on $[a,b]$ if, the sequence $s_n(x) = \sum_{k=1}^{n} f_k(x)$ of its partial sums converges uniformly to $s(x)$ on $[a,b]$.

Comparison of point-wise and uniform convergence

In a point-wise convergence, the choice of the index N depends both on the point chosen in $[a,b]$ and the $\varepsilon > 0$ and so we can write

$N = N(\varepsilon, x)$ This might, for example result in the following:

There might exist a sequence of points $x_1, x_2, x_3, \ldots \in [a,b]$

such that the sequence of indices $N(\varepsilon, x_1), N(\varepsilon, x_2), N(\varepsilon, x_3), \ldots$

is unbounded.

With a uniform convergence, this cannot occur since for each $\varepsilon > 0$

there exists a uniform N regardless of the choice of $x \in [a,b]$

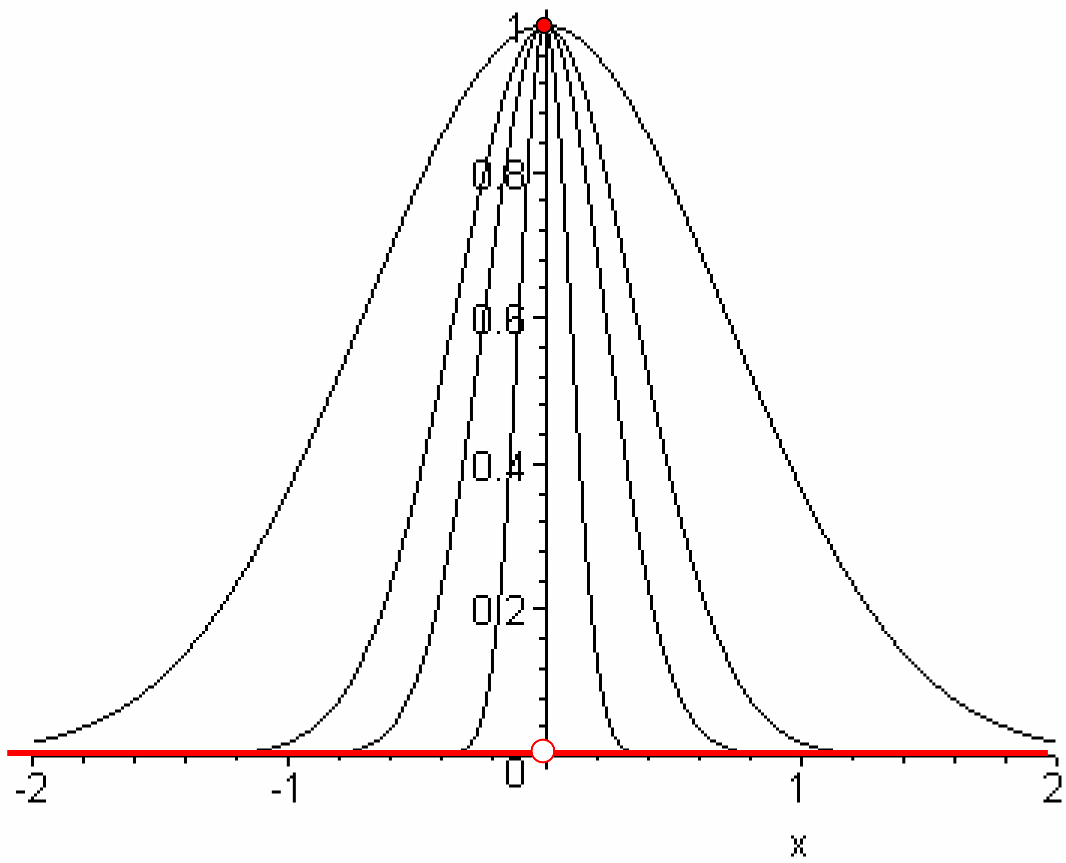
Cauchy's test of uniform convergence

Let a function sequence $\left\{ f_n\left(x\right)\right\}_{n=1}^{\infty}$ converge to a function $f\left(x\right)$.

It converges uniformly to $f\left(x\right)$ if and only if

$$\forall \varepsilon > 0 : \exists N : \forall m, n > N \wedge \forall x \in \left[a, b\right] : \left| f_m\left(x\right) - f_n\left(x\right)\right| < \varepsilon$$

Example $\{f_n(x)\}_{n=1}^{\infty} = \left\{e^{-nx^2}\right\}_{n=1}^{\infty}$ $f(x) = \begin{cases} 1 \text{ for } x = 0 \\ 0 \text{ otherwise} \end{cases}$



The above sequence does not converge uniformly. To prove this we will apply Cauchy's test and show that

$$\exists \varepsilon > 0 : \forall N : \exists m, n > N \wedge \exists x \in [a,b] : \left| e^{-mx^2} - e^{-nx^2} \right| \geq \varepsilon$$

Let $\varepsilon = \dfrac{1}{2}$ and m be an arbitrary index. Certainly, by letting x sufficiently close to zero so that, say, $x = x_0$, we will have

$$e^{-mx_0^2} > \frac{3}{4}$$

On the other hand, it is clear that if we chose an index $n > m$ sufficiently large we can "push this value down" so that

$$e^{-nx_0^2} < \frac{1}{4}$$

Then of course we have

$$\left| e^{-mx_0^2} - e^{-nx_0^2} \right| > \frac{1}{2}$$

Uniform convergence of sequences and series conserves some of the properties of the individual functions such as continuity and integrability.

Let a sequence $\left\{ f_n\left(x\right)\right\}_{n=1}^{\infty}$ of functions continuous on $[a,b]$ converge to a function $f(x)$. Let $\left\{ f_n\left(x\right)\right\}_{n=1}^{\infty}$ converge uniformly. Then $f(x)$ is continuous.

Let a sequence $\left\{ f_n\left(x\right)\right\}_{n=1}^{\infty}$ of functions integrable on $[a,b]$ converge to a function $f(x)$. Let $\left\{ f_n\left(x\right)\right\}_{n=1}^{\infty}$ converge uniformly. Then $f(x)$ is integrable and

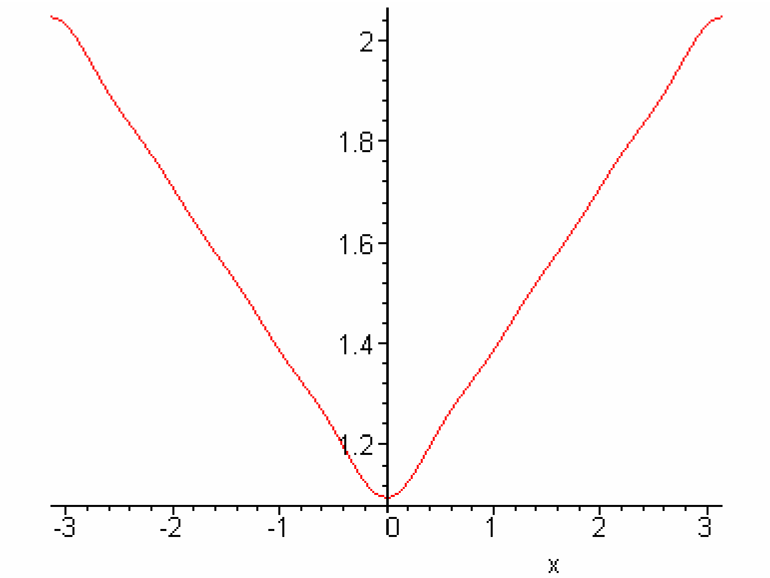
$$\int_a^b f\left(x\right) dx = \lim_{n\to\infty}\int_a^b f_n\left(x\right) dx$$

If all the functions of a sequence or a series are differentiable on $[a,b]$, it does not necessarily mean that so is their limit or sum as the example below shows:

The series $\frac{\pi}{2}-\frac{4}{\pi^2}\cos x-\frac{4}{9\pi^2}\cos 3x-\frac{4}{25\pi^2}\cos 5x-\frac{4}{49\pi^2}\cos 7x-\cdots$

has terms differentiable on $(-\pi,\pi)$ and converges uniformly to the function $y = |x|$, which is not differentiable at $x = 0$.

$$f := \frac{1}{2}\pi - \frac{4\cos(x)}{\pi^2} - \frac{4}{9}\frac{\cos(3x)}{\pi^2} - \frac{4}{25}\frac{\cos(5x)}{\pi^2} - \frac{4}{49}\frac{\cos(7x)}{\pi^2}$$



For the conservation of differentiation, only weaker theorems can be proved:

Let $\left\{ f_n\left(x\right)\right\}_{n=1}^{\infty}$ be a function sequence with every $f_n\left(x\right)$ having a continuous derivative in [a,b]. Let $\left\{ f_n\left(x\right)\right\}_{n=1}^{\infty}$ converge at at least one point $x_0 \in \left[a,b\right]$ and let the sequence $\left\{ \frac{d\ f_n\left(x\right)}{dx} \right\}_{n=1}^{\infty}$ converge uniformly in [a,b]. Then $\left\{ f_n\left(x\right)\right\}_{n=1}^{\infty}$ converges uniformly in [a,b], the limit $f\left(x\right)$ of this sequence is differentiable in [a,b] and

$$\frac{d\ f\left(x\right)}{dx} = \lim_{n\to\infty} \frac{d\ f_n\left(x\right)}{dx}$$

Let $\sum_{n=1}^{\infty} f_n(x)$ be a function series with every $f_n(x)$ having a continuous derivative in $[a,b]$. Let $\sum_{n=1}^{\infty} f_n(x)$ converge at at least one point $x_0 \in [a,b]$ and let the sequence $\sum_{n=1}^{\infty} \frac{d\ f_n(x)}{dx}$ converge uniformly in $[a,b]$. Then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly in $[a,b]$, $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is differentiable in $[a,b]$ and

$$\frac{d\ f(x)}{dx} = \sum_{n=1}^{\infty} \frac{d\ f_n(x)}{dx}$$

Uniform convergence test for series (Weierstrass)

Let $\sum_{n=1}^{\infty} f_n(x)$ be a series of functions defined on $[a,b]$.

If $\left|f_n(x)\right| \leq a_n$ for every n and the number series $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ uniformly converges.

Example

The Weierstrass test can be used to prove the uniform convergence of the series from the previous example:

$$\frac{\pi}{2}-\frac{4}{\pi^2}\cos x-\frac{4}{9\pi^2}\cos 3x-\frac{4}{25\pi^2}\cos 5x-\frac{4}{49\pi^2}\cos 7x-\cdots\le$$

$$\le\frac{\pi}{2}-\frac{4}{\pi^2}\left(\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots\right)$$

Now the series in brackets is selected from the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ whose convergence can be proved using, for example, the integral criterion.

<u>Uniform convergence test for series (Dirichlet)</u>

A series $\sum_{n=1}^{\infty} a_n f_n(x)$ uniformly converges in [a,b] if $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence converging to zero and if the partial sums of $\sum_{n=1}^{\infty} f_n(x)$ are uniformly bounded, that is, if, for every $x \in [a,b]$ and every n, we have $\sum_{n=1}^{n} f_n(x) \leq M$ where $M > 0$.