Power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + a_3 (x-x_0)^3 + \cdots$$

is called a power series with the centre at x_0 or centred around x_0

A power series centred around zero is a special case:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

When does a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ define a function?

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

In other words, for which x the resulting number series

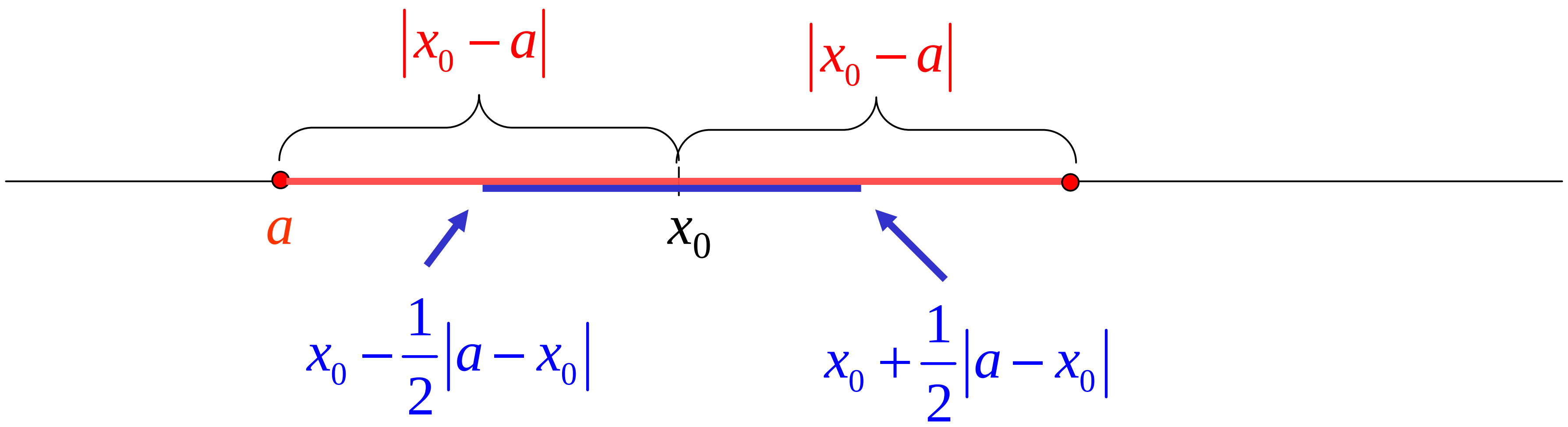
$\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges ?

It certainly does at $x = x_0$ since

$$\sum_{n=0}^{\infty} a_n (x_0 - x_0)^n = \sum_{n=0}^{\infty} a_n 0 = 0$$

If a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges at a point $a \neq x_0$, then it converges absolutely in the open interval $\left(x_0 - |x_0 - a|, x_0 - |x_0 + a|\right)$ and converges uniformly in each closed interval

$\left[x_0 - \vartheta|x_0 - a|, x_0 + \vartheta|x_0 - a|\right]$ where $0 < \vartheta < 1$



Unless a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges at any real number, a number $r > 0$ exists such that $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges absolutely for each x such that $|x - x_0| < r$ and diverges for any other x

This number is called the radius of convergence and $(x_0 - r, x_0 + r)$ is the interval of convergence of $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

Note: if a power series only converges at its center, we put $r = 0$.

Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ be a power series and let $\lambda = \limsup \sqrt[n]{|a_n|}$

Then for the radius of convergence r of $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ we have

$$r = \begin{cases} 0 & \text{for} \quad \lambda = \infty \\ \dfrac{1}{\lambda} & \text{for} \quad \lambda \neq \infty \\ \infty & \text{for} \quad \lambda = 0 \end{cases}$$

Note that if a_n has a limit, then $\lambda = \limsup a_n = \lim a_n$

Example

What is the radius of convergence of the power series $\sum \frac{x^n}{n}$?

$$\lim \sqrt[n]{n} = \lim e^{\frac{\ln n}{n}} = \lim e^{\frac{\ln n - \ln(n-1)}{n-(n-1)}} = \lim e^{\frac{\ln \frac{n}{n-1}}{1}} = e^0 = 1$$

Example

What is the radius of convergence of the power series $\sum \frac{x^n}{n!}$?

Let k be an arbitrary natural number. Then, for $n > k$ we have

$n! > k!k^{n-k}$. This means that $\sqrt[n]{n!} > \sqrt[n]{k!}\sqrt[n]{k^{n-k}} = \sqrt[n]{k!}k^{1-\frac{k}{n}}$

Since $\lim \sqrt[n]{k!}k^{1-\frac{k}{n}} = 1 \cdot k = k$, we have $\lim \sqrt[n]{n!} > k$ for any k,

which means that $\lim \sqrt[n]{n!} = \infty$ and $\lambda = \lim \frac{1}{\sqrt[n]{n!}} = 0$

Thus $r = \infty$

Derivative of a power series

If we differentiate each term of the power series $\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$,

we obtain a new power series $\sum_{n=0}^{\infty} n a_n \left(x - x_0\right)^{n-1}$

This new power series is then called the derivative of a given power series.

The power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \cdots$$

and its derivative

$$\sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} = 0 + a_1 + 2a_2 (x - x_0) + 3a_3 (x - x_0)^2 + \cdots$$

have the same radius of convergence

If $\sum_{n=0}^{\infty} a_n (x-x_0)^n = s(x)$ then $\sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} = \frac{ds(x)}{dx}$

for $x \in [-\vartheta r, \vartheta r]$ where r is the radius of convergence of

both the power series and $0 < \vartheta < 1$

Since $\sum_{n=1}^{\infty} n a_n \left(x - x_0\right)^{n-1}$ is again a power series with the identical radius of convergence r it has again a derivative in the convergence interval $(-r, r)$. This derivative is again a power series with the same radius of convergence r and so on:

$s(x) = \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$ has derivatives of arbitrary order in its convergence interval $(-r, r)$

$$\boxed{s^{(k)}(x) = \sum_{n=0}^{\infty} k!\binom{n}{k} a_n \left(x - x_0\right)^{n-k}}$$

Example

We know that $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$ for $-1 < x < 1$

Differentiating the series term by term yields:

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

$$\vdots$$

$$k! + \frac{(k+1)!}{1!}x + \frac{(k+2)!}{2!}x^2 + \frac{(k+3)!}{3!}x^3 + \cdots = \frac{k!}{(1-x)^{k+1}}$$

Integral of a power series

Let $\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n = s\left(x\right)$ be a power series with r as the radius of convergence. We can integrate it term by term from 1 to x where $\left|x\right| < r$ obtaining

$$\int_0^x s\left(t\right) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}\left(x - x_0\right)^{n+1}$$

The radius of convergence of the resulting series is again r.

Example

If we integrate term by term the power series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^{n-1} x^{n-1} + \cdots$$

from 0 to x where $x \in (-1,1)$, we obtain the power series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$$

Note that the resulting power series converges even for $x = 1$ but not for x = –1.

Example

If we integrate term by term the power series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \cdots + (-1)^{n-1} x^{2(n-1)} + \cdots$$

from 0 to x where $x \in (-1,1)$, we obtain the power series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \cdots$$

Note that the resulting power series converges even for $x = -1$ and $x = 1$.

Calculating π

The preceding formula can be used for calculating numerically the constant π

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots$$

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{n-1}\frac{1}{2n-1} + \cdots$$

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots + (-1)^{n-1}\frac{4}{2n-1} + \cdots$$

However, this series converges very slowly and is therefore not at all suitable for numerical calculation. To reach a reasonable accuracy of several decimal places probably several billions terms would have to be calculated.

The reason is that we have to use x at the end of the convergence interval. For practical calculations, better schemes have been devised such as:

$$\pi = 16 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right)$$

This identity was discovered by Machin in the 18th century.

$$\pi = 16\left(\frac{1}{5} - \frac{1}{5^3 \cdot 3} + \frac{1}{5^5 \cdot 5} - \frac{1}{5^7 \cdot 7} + \frac{1}{5^9 \cdot 9} - \frac{1}{5^{11} \cdot 11} + \cdots\right) +$$

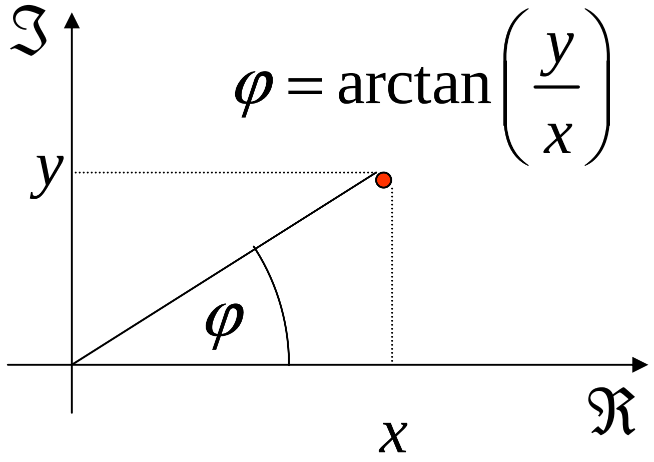
$$+4\left(\frac{1}{239} - \frac{1}{239^3 \cdot 3} + \frac{1}{239^5 \cdot 5} - \frac{1}{239^7 \cdot 7} + \frac{1}{239^9 \cdot 9} - \frac{1}{239^{11} \cdot 11} + \cdots\right) =$$

$$= 3.141\,592\,652\,615\ldots$$

$$3.141\,592\,653\,589\ldots$$

pocket calculator value

How Machin's identity can be derived:



$$\varphi = \arctan\left(\frac{y}{x}\right)$$

$$\arctan\left(\frac{y}{x}\right) = \Im \ln\left(x+iy\right)$$

$$\ln\left(x+iy\right) = \ln\sqrt{x^2+y^2} + i\varphi$$

$$\frac{\left(5+i\right)^4}{239+i} = 2+2i$$

This can be verified by calculation

$$\ln\left(\frac{\left(5+i\right)^4}{239+i}\right) = \ln\left(2+2i\right) \Rightarrow 4\ln\left(5+i\right) - \ln\left(239+i\right) = \ln\left(2+2i\right)$$

$$4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) = \arctan\left(\frac{2}{2}\right) = \frac{\pi}{4}$$