Let a function $f(x)$ be given as the sum of a power series in the convergence interval of the power series $f(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n$

Then such a power series is unique and its coefficients are given by the formula $a_n = \dfrac{f^{(n)}(x_0)}{n!}$

If a function $f(x)$ has derivatives of all orders at x_0, then we can formally write the corresponding Taylor series

$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

The power series created in this way is then called the Taylor series of the function $f(x)$. A Taylor series for $x_0 = 0$ is called MacLaurin series.

There are functions $f(\mathrm{x})$

whose formally generated Taylor series do not converge to it.

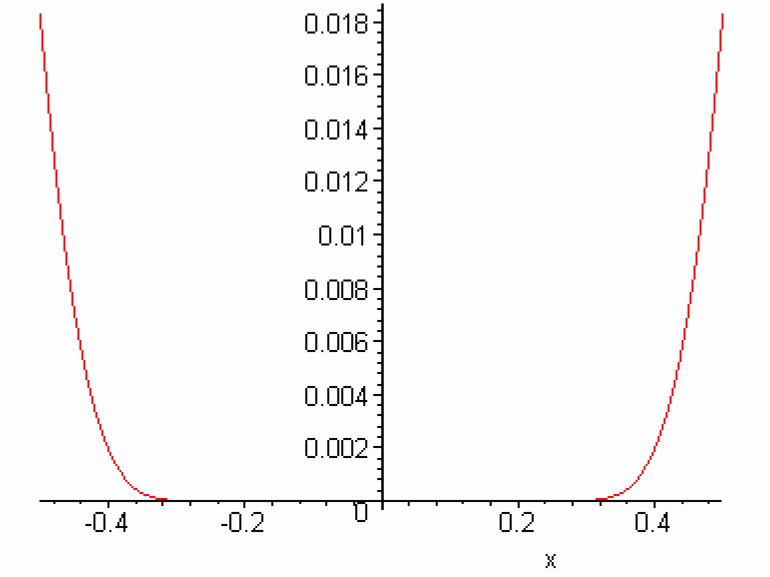
A condition that guarantees that **this will not happen** says that

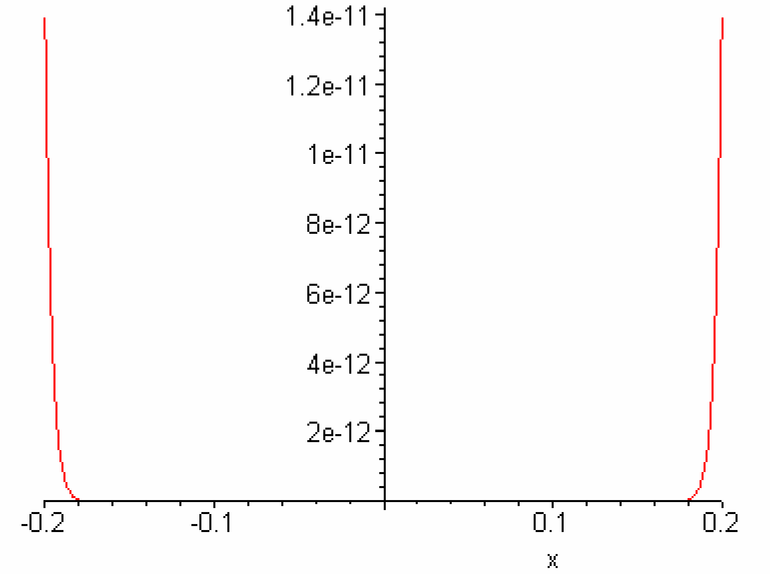
the derivatives of $f(x)$ are all uniformly bounded

in a neighbourhood of x_0.

There are functions with a Taylor series that, as a power series, converges to quite a different function as the following example shows:

Example $f(x) = e^{-\frac{1}{x^2}}$ for $x \neq 0$, $f(0) = 0$





For $x \neq 0$, we have

$$f'(x) = \frac{d\left(e^{-\frac{1}{x^2}}\right)}{dx} = \frac{2}{x^3} e^{-\frac{1}{x^2}} = \frac{2}{x^3 e^{\frac{1}{x^2}}}$$

and for $x = 0$:

$$f'(x) = \lim_{x \to 0} \frac{e^{-\frac{1}{x^2}} - 0}{x} = \lim_{x \to 0} \frac{1}{x e^{\frac{1}{x^2}}} = \lim_{x \to 0} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} = \lim_{t \to \infty} \frac{t}{e^{t^2}} = \lim_{t \to \infty} \frac{1}{2t e^{t^2}} = 0$$

In a similar way, we could also show that

$$0 = f(0) = f'(0) = f''(0) = \cdots = f^{(k)}(0) = \cdots$$

This means that the Taylor series corresponding to $f(x)$

converges to a constant function that is equal to zero at all

points. But clearly, $e^{-\frac{1}{x^2}} \neq 0$ for any $x \neq 0$.

Taylor series of some functions:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$