Function of a random variable

Let X be a random variable in a probabilistic space $(\Omega, \mathcal{S}, P)$ with a probability distribution $F(x)$

Sometimes we may be interested in another random variable Y that is a function of X, that is, $Y = g(X)$. The question is whether we can establish the probability distribution $G(y)$ of Y.

Let $g(x)$ be increasing on $R(X)$.

Since $F(x)$ is the probability distribution of X and $G(y)$ of Y, we can write

$$F(a) = P(X < a) \quad \text{and} \quad G(b) = P(Y < b) = P(g(X) < b)$$

$g(x)$ is increasing and so it has an inverse $g^{-1}(x)$

$$P(g(X) < b) = P\left(g^{-1}(g(X)) < g^{-1}(b)\right) = P\left(X < g^{-1}(b)\right) = F\left(g^{-1}(b)\right)$$

This means that $G(y) = F\left(g^{-1}(y)\right)$

If $g(x)$ is decreasing on $R(X)$, we can proceed in a similar way, but since the inverse of a decreasing function is again decreasing, the inequality will revert the relation sign and so

$$P(g(X)<b)=P\left(g^{-1}(g(X))\geq g^{-1}(b)\right)=$$

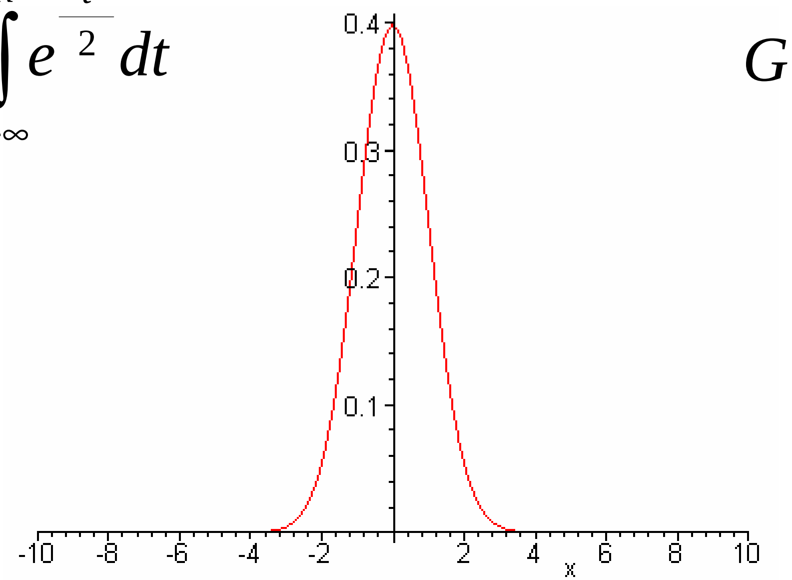
$$=P\left(X\geq g^{-1}(b)\right)=1-P\left(X<g^{-1}(b)\right)=1-F\left(g^{-1}(b)\right)$$

so that $G(y)=1-F\left(g^{-1}(y)\right)$

Sometimes we can use even more sophisticated methods as shown in the following example.

Random variable X has the standardized normal distribution $\Phi(x)$, calculate the distribution of the random variable $Y = X^2$.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt$$



$$G(y) = ?$$

$$G(y) = P\left(X^2 < y\right) = P\left(-\sqrt{y} < X < \sqrt{y}\right) = \Phi\left(\sqrt{y}\right) - \Phi\left(-\sqrt{y}\right)$$

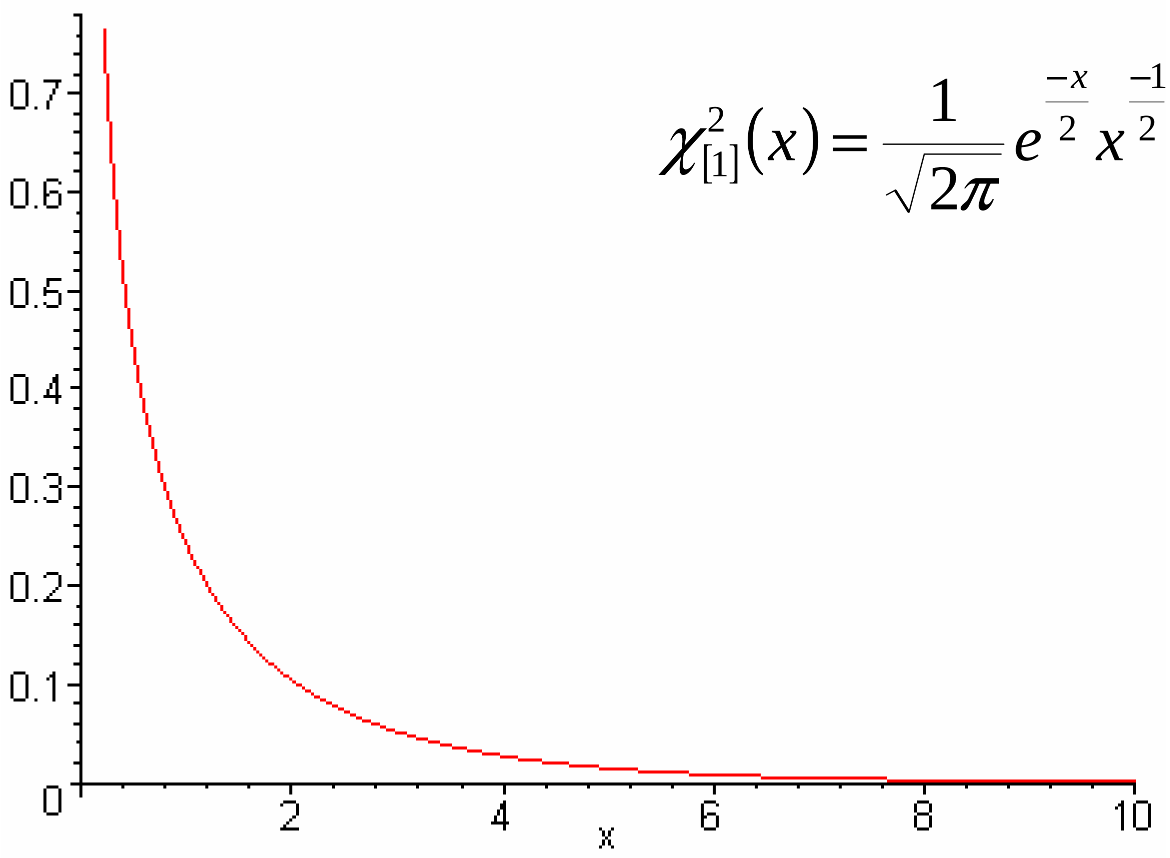
$$\Phi\left(\sqrt{y}\right) - \Phi\left(-\sqrt{y}\right) = \Phi\left(\sqrt{y}\right) - \left(1 - \Phi\left(\sqrt{y}\right)\right) = 2\Phi\left(\sqrt{y}\right) - 1$$

$$G(y) = \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{y}} e^{\frac{-t^2}{2}}\, dt - 1$$

We cannot calculate the last integral exactly, but we can establish the probability density of Y by differentiating $G(y)$

$$g(y) = \frac{dG(y)}{dy} = \frac{2}{\sqrt{2\pi}} e^{\frac{-y}{2}} \frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}} e^{\frac{-y}{2}} y^{\frac{-1}{2}}$$

This is the probability density of the random variable X^2. We have, in fact, calculated the density of the chi-squared distribution with one degree of freedom.



$$\chi^2_{[1]}(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x}{2}} x^{\frac{-1}{2}}$$

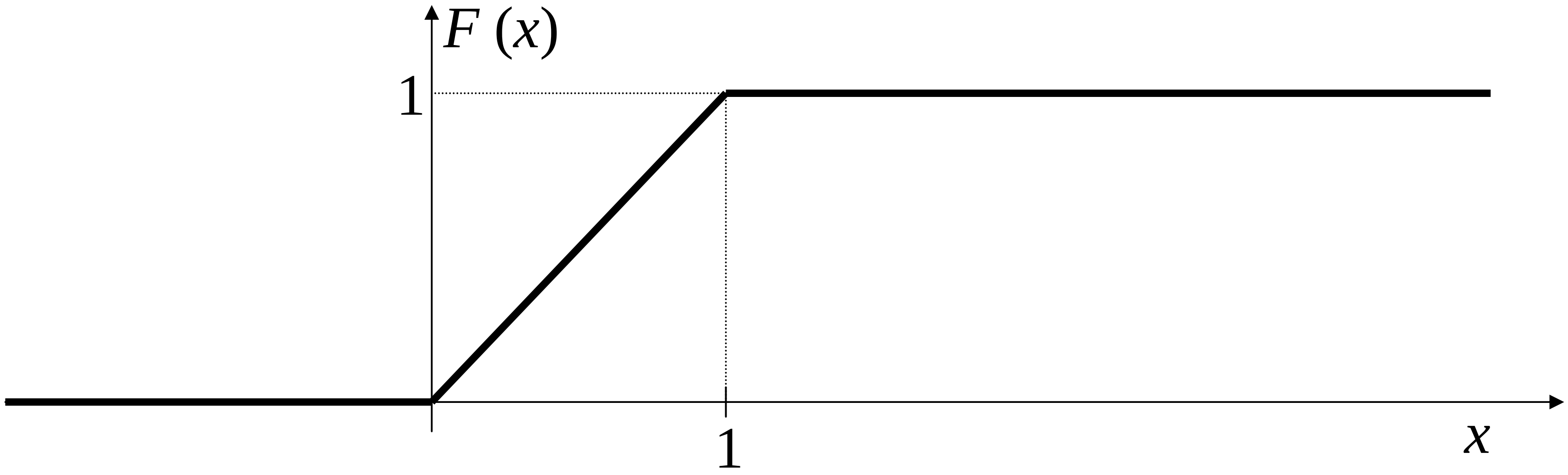
Example

Random variable X has a uniform distribution on [0,1]. Find a transformation $g(X)$ such that $Y = g(X)$ has a distribution $F(y)$.

X has the density $f(x) = 1$ for $x \in [0,1)$, $f(x) = 0$ otherwise

and the distribution

$F(x) = 0$ for $x < 0$, $F(x) = x$ for $x \in [0,1)$, $F(x) = 1$ for $x \geq 1$



$F(x)$ is a distribution and as such has R as the domain and [0,1] as the range. This means that, if $F(x)$ is increasing, it has an inverse $F^{-1}(x)$ that is also increasing with range R and domain [0,1].

Consider the transformation $Y = F^{-1}(x)$. We have

$$P(Y < a) = P\left(F^{-1}(X) < a\right) = P\left(F\left(F^{-1}(X)\right) < F(a)\right) =$$

$$= P\left(X < F(a)\right) = F(a)$$

Thus, we have shown that $F(y)$ is the distribution of $Y = F^{-1}(x)$.

This can be used for example for simulating a distribution using a pocket calculator.