Let $X_1, X_2, \ldots, X_n$ be independent random variables each having the standardized normal distribution $N(0,1)$, then the random variable

$$Y = X_1^2 + X_2^2 + \cdots + X_n^2$$

has a **<u>chi-squared distribution</u>** with n degrees of freedom

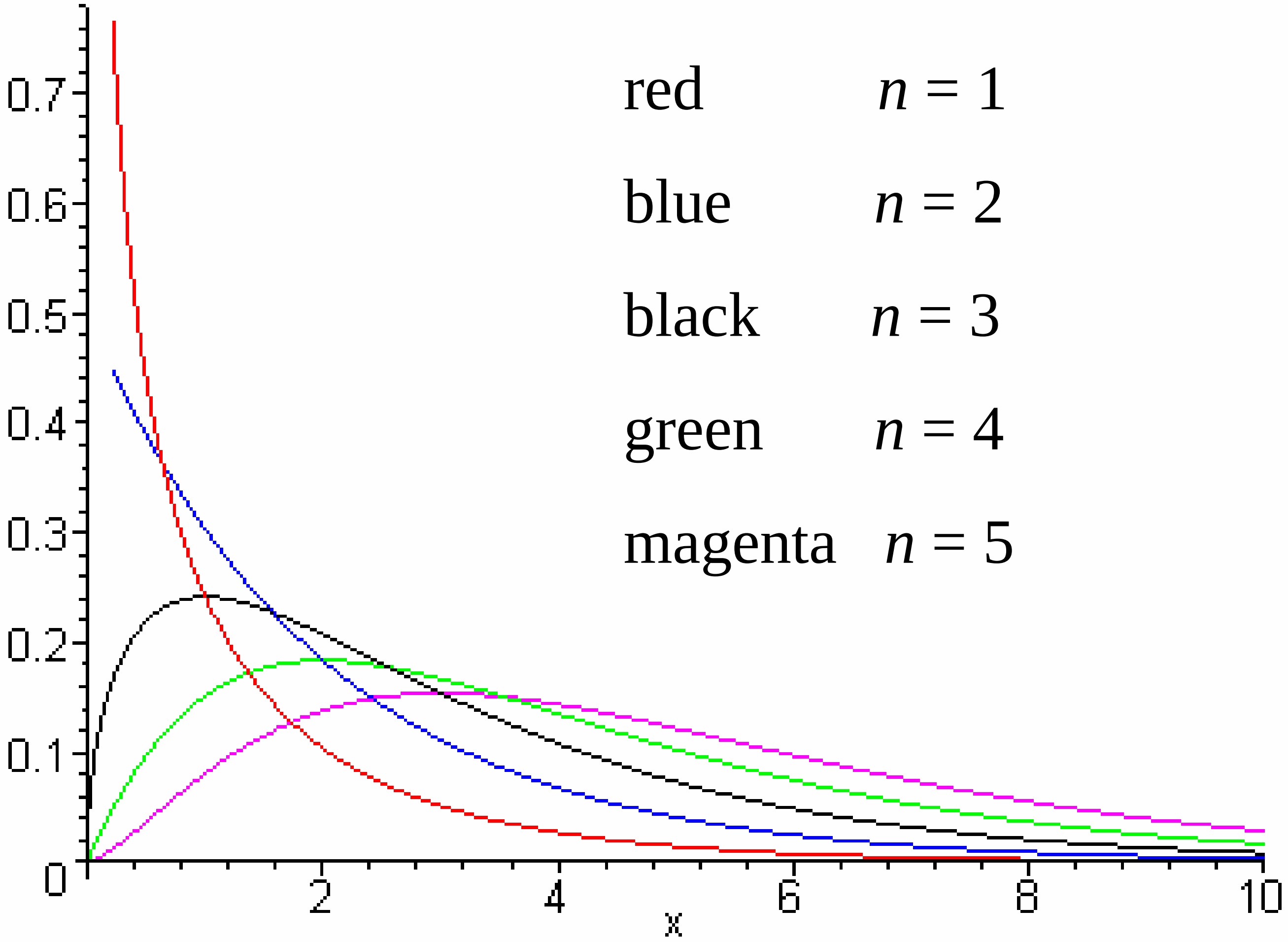
denoted by χ_n^2 with the probability density defined as

$$f_n(y) = \frac{1}{2^{n/2}\Gamma(n/2)} y^{n/2-1} e^{-y/2} \qquad \text{for } y > 0$$

$$f_n(y) = 0 \qquad \text{for } y \le 0$$

$$E\left(\chi_k^2\right) = k \qquad D\left(\chi_k^2\right) = 2k$$

For $k > 2$, χ_k^2 reaches its maximum at $k - 2$



We first prove this for $n = 1$, that is, $X \sim \chi_1^2$

Put $Y = X^2$. The distribution $G(y)$ of Y for $y \le 0$ equals zero and, for y > 0, we have

$$G(y) = P(Y < y) = P(X^2 < y) = P(-\sqrt{y} < X < \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t/2} dt.$ Thus

$$g(y) = G'(y) = \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{2\sqrt{y}} - \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{-1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}} e^{-y/2} y^{-1/2}$$

Next we shall use induction by n.

Let us have for some *k*:

$$f_k(y) = \frac{1}{2^{k/2}\Gamma(k/2)} y^{k/2-1} e^{-y/2} \qquad \text{for } \ y > 0.$$

$$f_1(y) = \frac{1}{2^{1/2}\Gamma(1/2)} y^{-1/2} e^{-y/2}$$

We can now use the following assertion in the theory of probability:

If (X, Y) is a random vector with a density $p(x,y)$, then the random variable $Z = X + Y$ has the density

$$f(z) = \int_{-\infty}^{\infty} p(u, z-u)\, du$$

Since $X_1^2 + X_2^2 + \cdots + X_k^2$ and X_{k+1}^2 are independent, we have

$$f(z) = \int_{-\infty}^{\infty} f_k(u-z) f_1(z)\, dz$$

$$f(z) = \frac{1}{2^{k/2}\Gamma(k/2)} \frac{1}{2^{1/2}\Gamma(1/2)} \int_0^y (y-z)^{k/2-1} z^{-1/2} e^{-\frac{y-z}{2}} e^{-\frac{z}{2}} dz$$

We use a transformation $z = yu$ with the Jacobian determinant $J = y$

$$f(z) = \frac{1}{2^{(k+1)/2}\Gamma(k/2)\Gamma(1/2)} \int_0^1 (y-uy)^{k/2-1} u^{-1/2} y^{-1/2} e^{-\frac{y-uy}{2}} e^{-\frac{uy}{2}} y\,du =$$

$$= \frac{1}{2^{(k+1)/2}\Gamma(k/2)\Gamma(1/2)} y^{(k+1)/2-1} e^{-\frac{y}{2}} \int_0^1 (1-u)^{k/2-1} u^{-1/2} du =$$

$$= \frac{1}{2^{(k+1)/2}\Gamma(k/2)\Gamma(1/2)} y^{(k+1)/2-1} e^{-\frac{y}{2}} B(1/2, k/2) =$$

$$= \frac{B(1/2, k/2)}{2^{(k+1)/2}\Gamma(k/2)\Gamma(1/2)} y^{(k+1)/2-1} e^{-\frac{y}{2}} =$$

$$= \frac{1}{2^{(k+1)/2}\Gamma((k+1)/2)} y^{(k+1)/2-1} e^{-\frac{y}{2}}$$

Let X and Z be independent random variables such that

$X \sim N(0,1)$ and $Z \sim \chi_k^2$ then the random variable

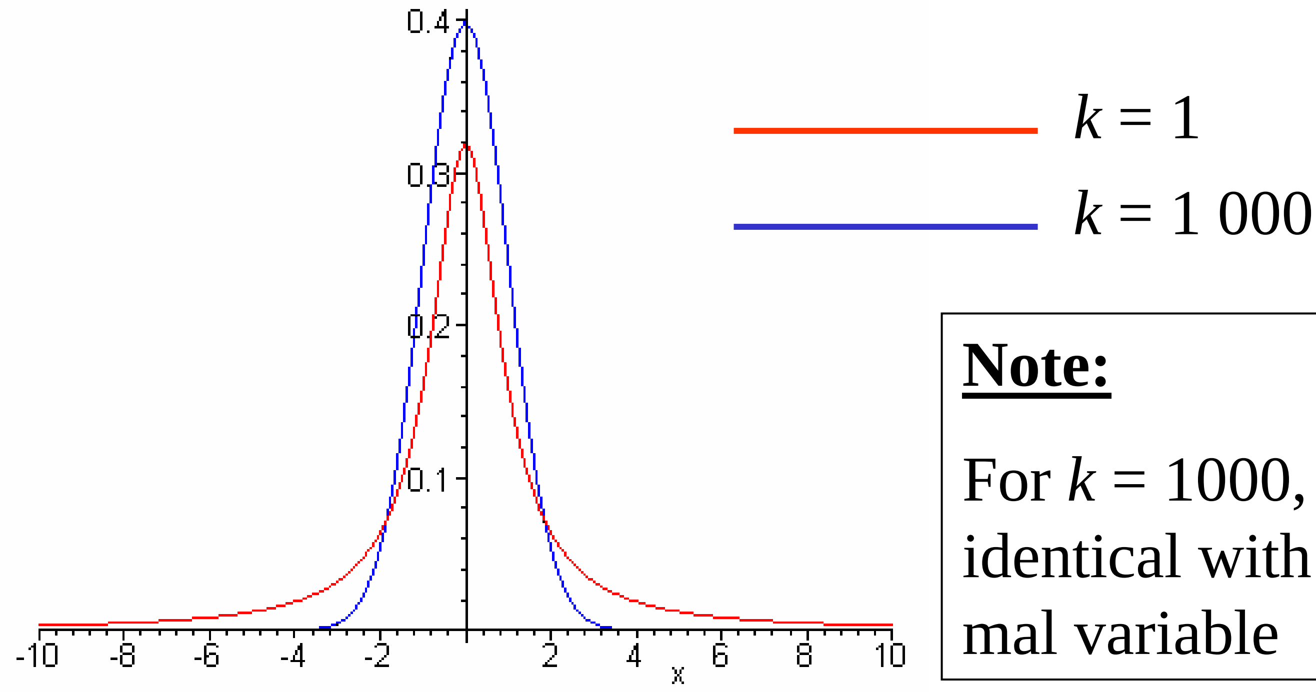
$$T = \frac{X}{\sqrt{Z/k}}$$

has **Student's *t*-distribution** with *k* degrees of freedom. This distribution is denoted by t_k and has the following probability density

$$f_k(t) = \frac{\Gamma((k+1)/2)}{\Gamma(k/2)\sqrt{\pi k}}\left(1+\frac{t^2}{k}\right)^{-(k+1)/2}, \quad -\infty < t < \infty$$

Probability density of $t_{[k]}$

$$S_{[k]} = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{x^2}{k}\right)^{-\frac{k+1}{2}}$$



Note:

For $k = 1000$, $t_{[k]}$ is practically identical with standardized normal variable

Expectancy and variance of $t_{[k]}$

Except for $k = 1$, the expectancy of $t_{[k]}$ is 0

For $k = 1, 2$ the expectancy and variance do not exist

For $k > 2$ we have $D(t_k) = \dfrac{k}{k-2}$

Derivation of the density for t_k

Since X and Z are independent, their simultaneous density is

$$g(x,z)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\frac{1}{2^{k/2}\Gamma(k/2)}z^{k/2-1}e^{-z/2}, \quad x\in R, z>0$$

Performing a transformation

$$T=\frac{X}{\sqrt{Z/k}}, \quad U=Z$$

with the inverse $X=T\sqrt{U/k}, \quad Z=U$ and Jacobian determinant

$J=\sqrt{u/k}$, the simultaneous density $h(t,u)$ can be written as

$$h(t,u) = \frac{1}{\sqrt{2\pi}} e^{-t^2 u/2k} \frac{1}{2^{k/2}\Gamma(k/2)} u^{k/2-1} e^{-u/2} \sqrt{u/k}, \quad t \in R, u > 0$$

After some simplification, we obtain

$$h(t,u) = \frac{1}{\sqrt{2\pi k}\, 2^{k/2}\Gamma(k/2)} u^{(k-1)/2} e^{-\frac{u}{2}\left(1+\frac{t^2}{k}\right)}, \quad t \in R, u > 0$$

Integrating by u, the density $f_k(t)$ of t_k can now be obtained as the marginal density of $h(t,u)$

We will use a transformation

$$u\left(1+\frac{t^2}{k}\right)=y \quad \text{with} \quad du=\frac{1}{\left(1+\frac{t^2}{k}\right)}dy$$

$$f_k(t)=\int_0^\infty \frac{1}{\sqrt{2\pi k}\,2^{k/2}\Gamma(k/2)}\frac{y^{(k-1)/2}}{\left(1+\frac{t^2}{k}\right)^{(k-1)/2}}e^{-y}\,\frac{2}{1+\frac{t^2}{k}}dy=$$

$$=\frac{1}{\sqrt{2\pi k}\,2^{k/2}\Gamma(k/2)}\frac{1}{\left(1+\frac{t^2}{k}\right)^{(k+1)/2}}\int_0^\infty y^{(k-1)/2}e^{-y/2}dy=$$

$$= \frac{\Gamma\left(k+1/2\right)}{\sqrt{\pi k}\,\Gamma\left(k/2\right)}\left(1+\frac{t^2}{k}\right)^{-(k+1)/2} \frac{1}{2^{(k+1)/2}\,\Gamma\left(k+1/2\right)}\int_0^{\infty} y^{(k+1)/2-1} e^{-y/2}\,dy =$$

$$= \frac{\Gamma\left(k+1/2\right)}{\sqrt{\pi k}\,\Gamma\left(k/2\right)}\left(1+\frac{t^2}{k}\right)^{-(k+1)/2}$$

since the integral is of the density of χ^2_{k+1}

Let $X \sim \chi_m^2$ and $Y \sim \chi_n^2$ be independent random variables, then then the random variable $Z = \dfrac{X/m}{Y/n}$ has a Fisher-Snedecor distribution $F_{m,n}$ with m and n degrees of freedom with a density

$$f_{m,n}(x) = \frac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)} (m/n)^{m/2} x^{m/2-1} \left(1 + \frac{m}{n} x\right)^{-(m+n)/2}$$

for $x > 0$, otherwise zero.

Plot of $f_{m,n}(x)$ for $m = 15$, $n = 10$