Euclidean space

Euclidean space is the set of all *n*-tuples of real numbers, formally

$$E_n = \{[x_1, x_2, \ldots, x_n] \mid x_i \in R, i = 1, 2, \ldots, n\}$$

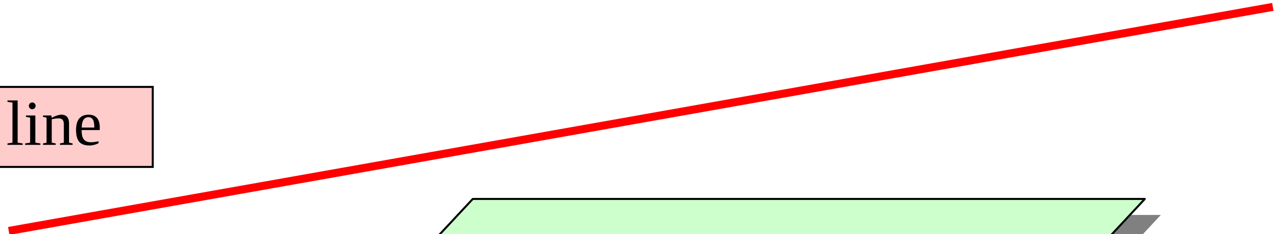
with a number called distance assigned to every pair of its

elements. Formally, if $X = [x_1, x_2, \ldots, x_n], Y = [y_1, y_2, \ldots, y_n]$ we

define

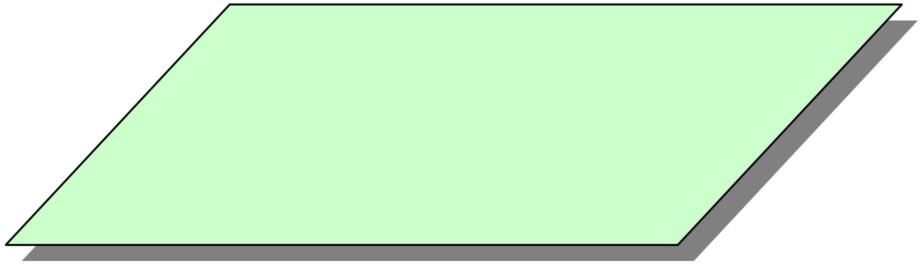
$$\rho(X, Y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

The elements of a Euclidean space E_n are sometimes referred to as points. Geometrically, the following identifications are done:



E_1 with a straight line

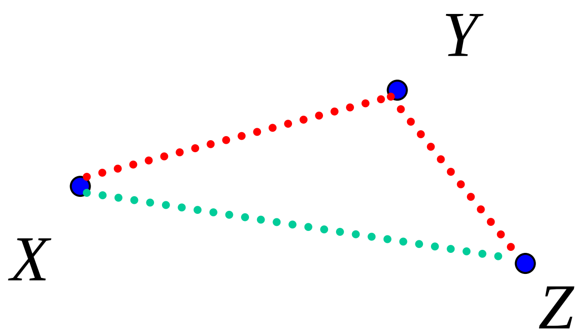
E_2 with a plane



E_3 with our 3-D space

The distance ρ has the following properties:

- $\rho(X,Y) \geq 0$
- $\rho(X,Y) = \rho(Y,X)$
- $\rho(X,Y) = 0 \Leftrightarrow X = Y$
- $\rho(X,Y) + \rho(Y,Z) \geq \rho(X,Z)$



The first three properties follow almost immediately, but the fourth one, which is called a triangle inequality, requires some effort:

In the sequel, we will assume that the summation indices always range from 1 to *n*. We will start with the following inequality, which is obvious:

$$\sum_{i<j}\left(a_i b_j - a_j b_i\right)^2 \geq 0$$

$$\sum_{i<j}\left(a_ib_j - a_jb_i\right)^2 \geq 0$$

$$\sum_{i<j}\left(a_i^2b_j^2 - 2a_ib_ja_jb_i + a_j^2b_i^2\right) \geq 0$$

$$\sum_{i\neq j}\left(a_i^2b_j^2 - 2a_ib_ja_jb_i\right) \geq 0$$

$$\sum_{i\neq j}a_i^2b_j^2 - 2\sum_{i\neq j}a_ib_ja_jb_i \geq 0$$

$$\sum_{i\neq j}a_i^2b_j^2 \geq 2\sum_{i\neq j}a_ib_ja_jb_i$$

$$\sum a_i^2b_i^2 + \sum_{i\neq j}a_i^2b_j^2 \geq \sum a_i^2b_i^2 + 2\sum_{i\neq j}a_ib_ja_jb_i$$

$$\sum a_i^2\sum b_i^2 \geq \left(\sum a_ib_i\right)^2$$

Hölder's inequality

$$\sum a_i^2 \sum b_i^2 \ge \left(\sum a_i b_i\right)^2$$

We will now use it to finish the proof of the triangle inequality, that is, $\rho(X,Y)+\rho(Y,Z)\ge\rho(X,Z)$. By definition:

$$\sqrt{\sum\left(x_i-y_i\right)^2}+\sqrt{\sum\left(y_i-z_i\right)^2}\ge\sqrt{\sum\left(x_i-z_i\right)^2}$$

Minkowski's inequality

$$\sum a_i^2 \sum b_i^2 \geq \left(\sum a_i b_i \right)^2$$

$$2\sqrt{\sum a_i^2 \sum b_i^2} \geq 2\sum a_i b_i$$

$$\sum a_i^2 + 2\sqrt{\sum a_i^2 \sum b_i^2} + \sum b_i^2 \geq \sum a_i^2 + 2\sum a_i b_i + \sum b_i^2$$

$$\left(\sqrt{\sum a_i^2} + \sqrt{\sum b_i^2} \right)^2 \geq \sum \left(a_i + b_i \right)^2$$

$$\sqrt{\sum a_i^2} + \sqrt{\sum b_i^2} \geq \sqrt{\sum \left(a_i + b_i \right)^2}$$

$$\sqrt{\sum a_i^2} + \sqrt{\sum b_i^2} \geq \sqrt{\sum (a_i + b_i)^2}$$

Substituting $a_i = x_i - y_i, b_i = y_i - z_i$ finally yields

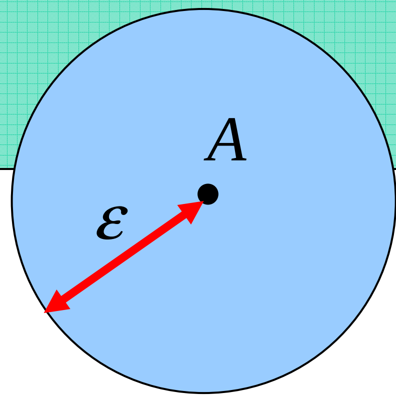
$$\sqrt{\sum (x_i - y_i)^2} + \sqrt{\sum (y_i - z_i)^2} \geq \sqrt{\sum (x_i - z_i)^2}$$

Neighbourhood of a point in E_n

If $A = [a_1, a_2, \ldots, a_n]$ is a point in E_n, then, for any $\varepsilon > 0$, the set

$$N(A, \varepsilon) = \{X \in E_n \mid \rho(A, X) < \varepsilon\}$$

is called an ε-neighbourhood of *A*



Open sets

Let $M \subseteq E_n$. If, for any $X \in M$ there is an $\varepsilon > 0$ such that

$$N(X, \varepsilon) \subseteq M$$

we say that M is an open set in E_n.

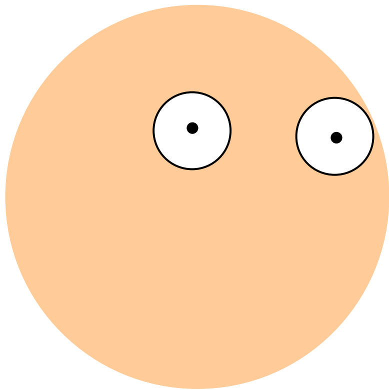
In other words, a set M is open if,

with every point, it contains at

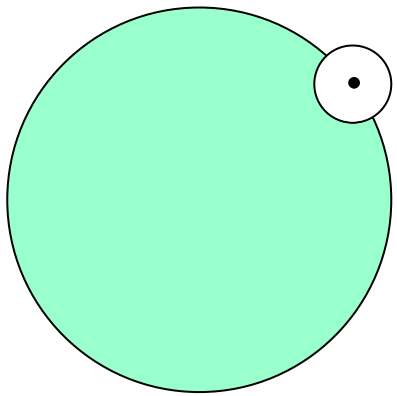
least one of its neighbourhoods.

Examples

$M \subseteq E_2$ given by the equation $x^2 + y^2 < 4$ is an open set.



whereas $C \subseteq E_2$ given by the equation $x^2 + y^2 \leq 4$ is not.



Complement

The complement of a set $M \subseteq E_n$ will be denoted by c(M).

It is the set of all the points in E_n that are not in M.

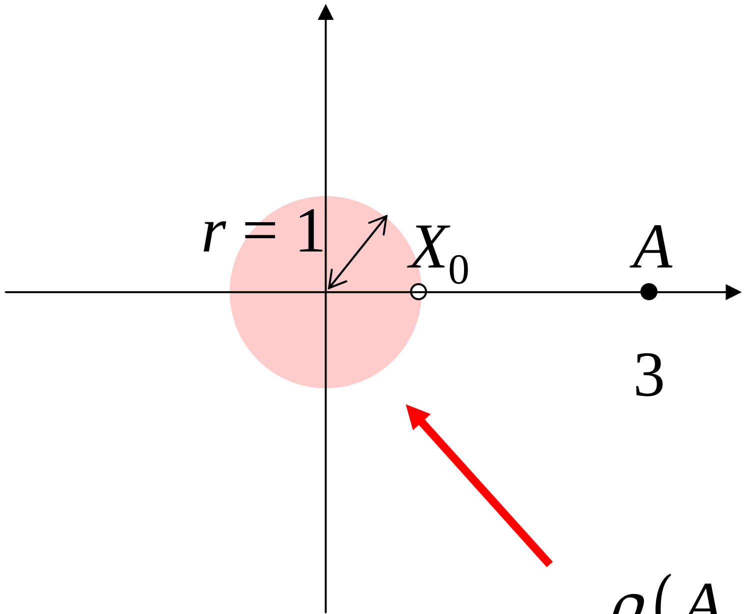
Distance of a point from a set

Let A be a point in E_n and $M \subseteq E_n$. We define the distance of A from M as the greatest lower bound of the distances of A from the points in M. Formally:

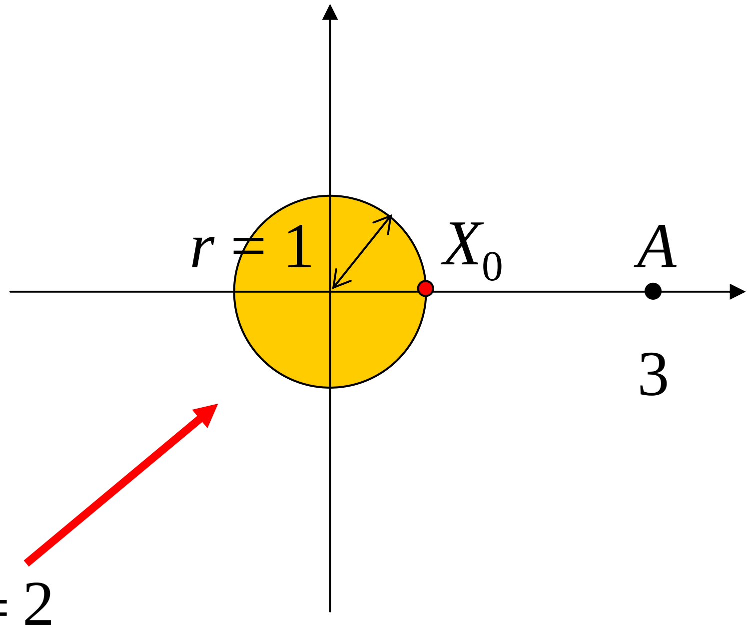
$$\rho(A, M) = \text{glb}\{\rho(A, X) \mid X \in M\}$$

Example

$M : x^2 + y^2 < 1$



$M : x^2 + y^2 \le 1$



$$\rho(A, M) = 2$$

The point at which the glb is realised is **not in *M***.

The point at which the glb is realised is **in *M***.

Distance of two sets

The distance of two sets $M, N \subseteq E_n$ is defined as the greatest lower bound of the distances between points in M and points in N. Formally

$$\rho(M, N) = \text{glb}\{\rho(x, y) \mid x \in M \wedge y \in N\}$$

Closure of a set

The closure $\overline{M}$ of a set $M \subseteq E_n$ is defined as the set of all the points in E_n such that their distance from M is zero.

Note that $\overline{M}$ will certainly contain all the points from M, but, in addition, may contain points not in M but close enough to it so that their distance from M is zero. Thus we have $M \subseteq \overline{M}$

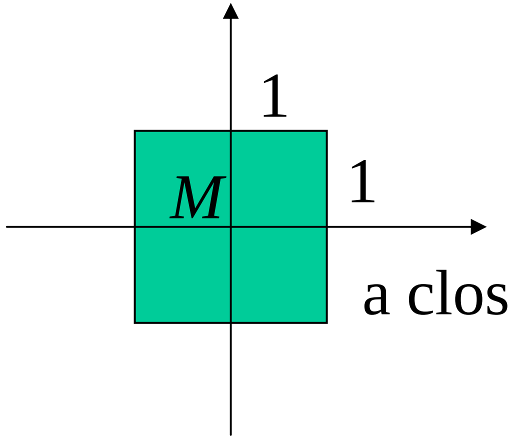
Closed sets

If, for an $M \subseteq E_n$, we have $\overline{M} = M$, we say that M is closed

Another equivalent definition of a closed set is that a set $M \subseteq E_n$ is closed if its complement $c(M)$ is open.

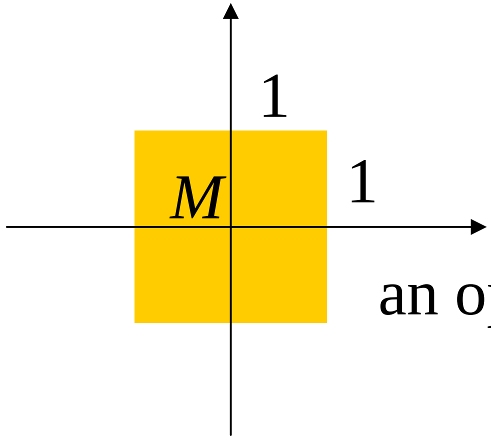
Examples

$$M : |x| \le 1 \land |y| \le 1$$



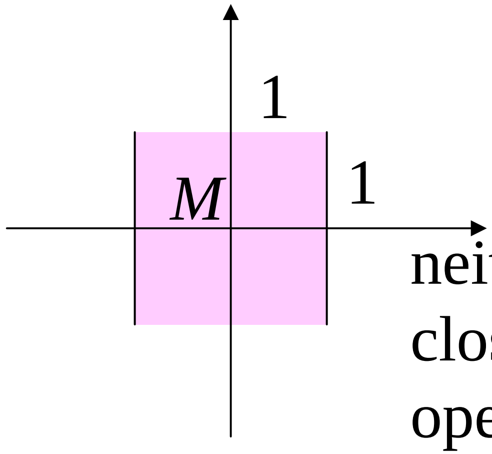
a closed set

$$M : |x| < 1 \land |y| < 1$$



an open set

$$M : |x| \le 1 \land |y| < 1$$



neither closed nor open set

Border of a set

The border B of a set $M \subseteq E_n$ is defined as the set of all the border points where a border point is a point whose every neighbourhood contains points both in M and outside M.

Another, more formal, definition may be the following:

$$B = \overline{M} \cap \overline{c(M)}$$

that is, B is composed precisely of those elements that lie both in the closure of M and the closure of its complement.

Interior of a set

The interior $i(M)$ of a set $M \subseteq E_n$ is defined as M without its border points. Formally, we can write $i(M) = M - \overline{c(M)}$

Finite unions and intersections

The union or intersection of a finite number of open sets is again an open set.

The union or intersection of a finite number of closed sets is again a closed set.

Unions of arbitrary classes of sets

The union of an arbitrary class (even infinite) of open sets

is again an open set

The union of an arbitrary (infinite) class of closed sets

may not be a closed set.

Example in E_1

If $M_i = \left[-1+\frac{1}{i}, 1-\frac{1}{i}\right]$ then $\bigcup_{i=1}^{\infty} M_i = (-1, 1)$

Indeed, for an $\varepsilon > 0$, clearly, an i exists such that

$-1+\varepsilon > -1+\frac{1}{i}$ and $1-\varepsilon < 1-\frac{1}{i}$ so that

$-1+\varepsilon \in M_i$ and $1-\varepsilon \in M_i \Rightarrow -1+\varepsilon \in \bigcup_{i=1}^{\infty} M_i$ and $1-\varepsilon \in \bigcup_{i=1}^{\infty} M_i$

On the other hand, neither 1 nor –1 are contained in any M_i

Intersections of arbitrary classes of sets

The intersection of an arbitrary class (even infinite) of closed sets is again a closed set

The intersection of an arbitrary (infinite) class of open sets may not be an open set.

Example in E_1

If $M_i = \left(-1-\frac{1}{i}, 1+\frac{1}{i}\right)$ then $\bigcap_{i=1}^{\infty} M_i = [-1,1]$

Clearly, any x such that $|x| \leq 1$ lies in all the sets M_i

On the other hand, for any $\varepsilon > 0$, we will certainly find an

index i_0 such that, for $i > i_0$, $-1-\varepsilon \notin M_i$ and $1+\varepsilon \notin M_i$