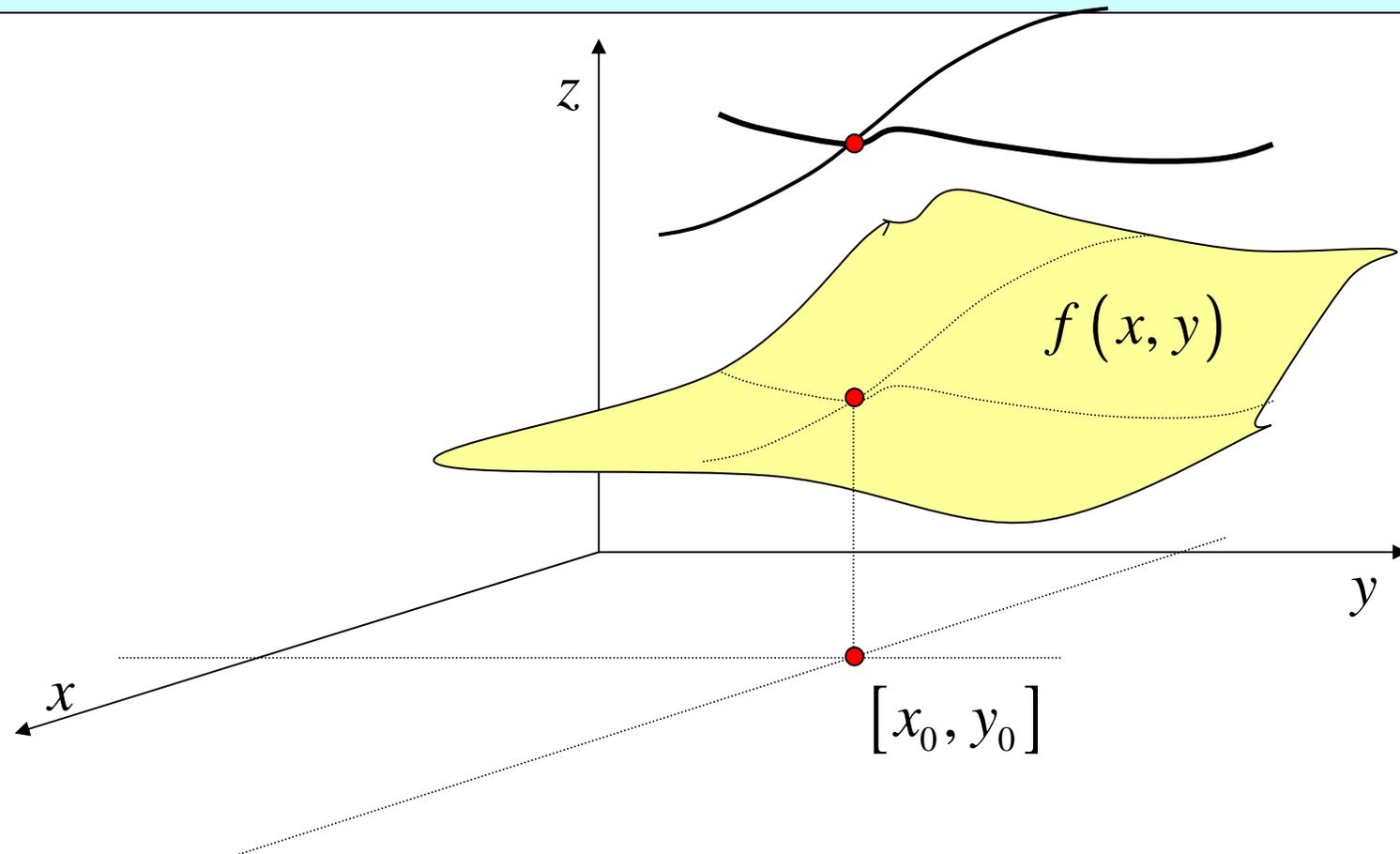


If a function of one variable is differentiable at a point, it is also continuous at this point. With functions of more variables, this implication is not so straightforward. There are functions with all partial derivatives at a point that are not continuous.



Let a function $f(X) = f(x_1, x_2, \dots, x_n)$ be given and let A be an internal point of its domain. Let us consider a function

$df_A(dx_1, dx_2, \dots, dx_n) = df_A(X)$ of n variables where $dx_i = x_i - a_i$

$$df_A(X) = \frac{\partial f(A)}{\partial x_1} dx_1 + \frac{\partial f(A)}{\partial x_2} dx_2 + \dots + \frac{\partial f(A)}{\partial x_n} dx_n$$

If $\lim_{X \rightarrow A} \frac{f(X) - f(A) - df_A(X)}{\rho(X, A)} = 0$ then the function $df_A(X)$

is called the **total differential** of $f(X)$ at A .

If a function $f(x_1, x_2, \dots, x_n)$, at some point $A \in D(f)$, has all the partial derivatives and if all these partial derivatives are continuous functions at A , then $f(x_1, x_2, \dots, x_n)$ has a total differential at A .

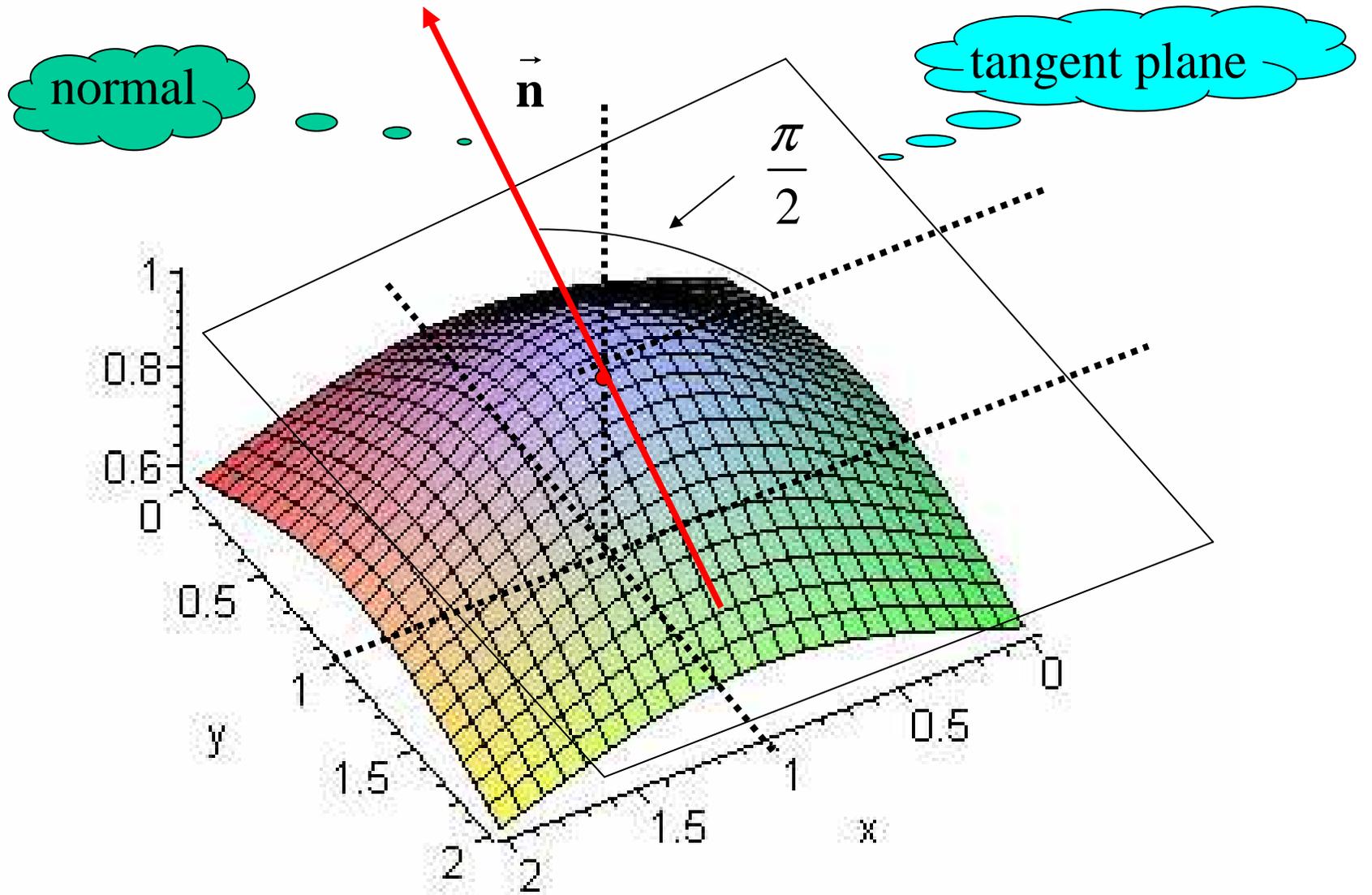
Instead of saying that a function has a total differential at a point A we can also say that it is differentiable at A .

If a function $f(x_1, x_2, \dots, x_n)$, at some point $A \in D(f)$, has a total differential, then $f(x_1, x_2, \dots, x_n)$ is continuous at A .

Tangent plane to the graph of a 2-function

If a function $f(x, y)$ is differentiable at a point $A = [a_1, a_2]$ its graph has a tangent plane at this point that can be defined by the formula:

$$z - f(a_1, a_2) = \frac{\partial f(a_1, a_2)}{\partial x}(x - a_1) + \frac{\partial f(a_1, a_2)}{\partial y}(y - a_1)$$



Equations of the normal to the graph of a function at a point

$$\text{If } f'_x(a_1, a_2) \cdot f'_y(a_1, a_2) \neq 0$$

$$A = [a_1, a_2]$$

$$\frac{x - a_1}{-f'_x(a_1, a_2)} = \frac{y - a_2}{-f'_y(a_1, a_2)} = z - f(a_1, a_2)$$

parametric
equations

$$x = a_1 + t \frac{\partial f(a_1, a_2)}{\partial x}$$

$$y = a_2 + t \frac{\partial f(a_1, a_2)}{\partial y}$$

$$z = f(a_1, a_2) + t$$

If we determine the total differential of a function $f(x_1, x_2, \dots, x_n)$ at every point of $D(f)$ where it exists, we have actually defined a function of $2n$ variables:

$$df(x_1, x_2, \dots, x_n, dx_1, dx_2, \dots, dx_n)$$

Now we can view dx_1, dx_2, \dots, dx_n as parameters and determine the total differential of the function df with regard to the variables x_1, x_2, \dots, x_n at every point at which it exists. In this way we have defined a total differential of the total differential $df(x_1, x_2, \dots, x_n)$ which is called a second total differential of $f(x_1, x_2, \dots, x_n)$ and so on.

Example

For a two-function $f(x, y)$ we have

$$df(x, y) = f'_x(x, y)dx + f'_y(x, y)dy$$

$$\frac{\partial df(x, y)}{\partial x} = f''_{xx}(x, y)dx + f''_{yx}(x, y)dy$$

$$\frac{\partial df(x, y)}{\partial y} = f''_{xy}(x, y)dx + f''_{yy}(x, y)dy$$

$$d(df(x, y)) = d^2f(x, y) = f''_{xx}dx^2 + (f''_{yx} + f''_{xy})dxdy + f''_{yy}dy^2$$

$$\text{or } d^2f(x, y) = f''_{xx}dx^2 + 2f''_{xy}dxdy + f''_{yy}dy^2 \text{ if } f''_{xy}(x, y) = f''_{yx}(x, y)$$

To calculate the n -th differential of $f(x_1, x_2, \dots, x_n)$ we can use the following formal expression

$$d^n f(x_1, x_2, \dots, x_n) = \left(\frac{\partial}{\partial x_1} dx_1 + \frac{\partial}{\partial x_2} dx_2 + \dots + \frac{\partial}{\partial x_n} dx_n \right)^n f(x_1, x_2, \dots, x_n)$$

Example Calculate the third differential of $w = f(x, y, z)$

$$d^3 f(x, y, z) = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right)^3 f(x, y, z)$$

$$\begin{aligned} \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right)^3 &= \frac{3!}{3!0!0!} \frac{\partial^3}{\partial x^3} dx^3 + \frac{3!}{2!1!0!} \frac{\partial^3}{\partial x^2 \partial y} dx^2 dy + \\ &+ \frac{3!}{2!0!1!} \frac{\partial^3}{\partial x^2 \partial z} dx^2 dz + \frac{3!}{1!2!0!} \frac{\partial^3}{\partial x \partial y^2} dx dy^2 + \frac{3!}{1!0!2!} \frac{\partial^3}{\partial x \partial z^2} dx dz^2 + \\ &\frac{3!}{1!1!1!} \frac{\partial^3}{\partial x \partial y \partial z} dx dy dz + \frac{3!}{0!1!2!} \frac{\partial^3}{\partial y \partial z^2} dy dz^2 + \frac{3!}{0!2!1!} \frac{\partial^3}{\partial y^2 \partial z} dy^2 dz + \\ &+ \frac{3!}{0!3!0!} \frac{\partial^3}{\partial y^3} dy^3 + \frac{3!}{0!0!3!} \frac{\partial^3}{\partial z^3} dz^3 \end{aligned}$$



$$\begin{aligned}
d^3 f(x, y, z) = & \frac{\partial^3 f(x, y, z)}{\partial x^3} dx^3 + 3 \frac{\partial^3 f(x, y, z)}{\partial x^2 \partial y} dx^2 dy + \\
& + 3 \frac{\partial^3 f(x, y, z)}{\partial x^2 \partial z} dx^2 dz + 3 \frac{\partial^3 f(x, y, z)}{\partial x \partial y^2} dx dy^2 + 3 \frac{\partial^3 f(x, y, z)}{\partial x \partial z^2} dx dz^2 + \\
& + 6 \frac{\partial^3 f(x, y, z)}{\partial x \partial y \partial z} dx dy dz + 3 \frac{\partial^3 f(x, y, z)}{\partial y \partial z^2} dy dz^2 + 3 \frac{\partial^3 f(x, y, z)}{\partial y^2 \partial z} dy^2 dz + \\
& + \frac{\partial^3 f(x, y, z)}{\partial y^3} dy^3 + \frac{\partial^3 f(x, y, z)}{\partial z^3} dz^3
\end{aligned}$$

Using a total differential find the value of $2.1^{3.2}$

We will calculate the total differential of the function

$z = x^y$ at $x = 2$ and $y = 3$ and then substitute $dx = 0.1$, $dy = 0.2$

$$2.1^{3.2} = z(2.1, 3.2) \approx z(2, 3) + dz|_{dx=0.1, dy=0.2}$$

$$\frac{\partial z}{\partial x} = yx^{y-1} \Rightarrow 3 \cdot 2^2 = 12$$

$$\frac{\partial z}{\partial y} = x^y \ln x \Rightarrow 2^3 \ln 2 \approx 5.54518$$

$$2.1^{3.2} \approx 8 + 12(0.1) + 5.54518(0.2) = 10.30904$$